ACADEMIC PRESS

Journal of Algebra 253 (2002) 237–275

JOURNAL OF Algebra

www.academicpress.com

q-centralizer algebras for spin groups

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Received 13 August 2001

Communicated by G.D. James

Abstract

Let V and V_ϵ be the vector and a spinor representation of $\mathfrak{so}_N$, $N \in \mathbb{N}$. We show that the quantum group $U_q(\mathfrak{so}_N)$ and the braid group corresponding to the Dynkin diagram B_f (the latter acting via R-matrices) are each others commutant on $V_\epsilon \otimes V^{\otimes f}$. Moreover, these braid representations factor through specializations of Häring–Oldenburg's B-BMW-algebra; this also holds with V_ϵ replaced by $V_{m\epsilon}$, $m \in \mathbb{N}$, if N is even. We use this observation to compute the weights of the Markov trace of this algebra as a 2-variable function, and the values of the parameters for which it is semisimple. © 2002 Elsevier Science (USA). All rights reserved.

Introduction

Let V be the vector representation of the Drinfeld–Jimbo quantum group $U_q\mathfrak{so}_{2n}$, and let $V_{m\epsilon}$ be the irreducible $U_q\mathfrak{so}_{2n}$ representation with highest weight $m\epsilon$, where ϵ is the highest weight of one of the spinor representations of $\mathfrak{so}_{2n}$ and $m \in \mathbb{N}$. In this paper we study the centralizer algebra $\mathcal{A}_f := \mathrm{End}_{U_q\mathfrak{so}_{2n}}(V_{m\epsilon} \otimes V^{\otimes f})$. We show that $\mathcal{A}_f$ is generated by the R-matrices $\check{R}_{V,V_{m\epsilon}}\check{R}_{V_{m\epsilon},V}$, acting on the first 2 factors of $V_{m\epsilon} \otimes V^{\otimes f}$, and $\check{R}_i$, acting as the

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1 Partially supported by NSF grant DMS 0070848.

0021-8693/02/$ – see front matter
PII: S0021-8693(02)00069-8

R-matrix $\check{R}_{V,V}$ on the $(i+1)$st and the $(i+2)$nd factor of $V_{m\epsilon} \otimes V^{\otimes f}$ (see Theorem 2.7). The methods for proving this result are well-known, and similar results have appeared before (see, e.g., [17,28]). We also show that in the classical limit $q = 1$, the centralizer of the $\mathfrak{so}_{2n}$-action is generated by Brauer's centralizer algebra and a projection onto an irreducible submodule of $V_{m\epsilon} \otimes V$. Similar results hold for $U_q\mathfrak{so}_{2n+1}$.

Our results imply that $\mathcal{A}_f$ is a quotient of a specialization of an algebra BB_f, depending on 3 parameters q, r and r', which has been defined by Häring-Oldenburg under the name B-BMW-algebra [9]. This observation allows us to simplify some of the proofs of results by Häring-Oldenburg. Moreover, we obtain explicit formulas for the weights of the Markov trace as a new result (see Theorem 4.7). More precisely, these are rational functions in the variables q, r, and r', labeled by pairs of Young diagrams; for any irreducible representation V_λ in $V_{m\epsilon} \otimes V^{\otimes f}$ the quotient $\dim_q(V_\lambda)/\dim_q V_{m\epsilon}$ is obtained for suitable specializations of the variables of our functions. This allows us to determine exactly for which values of the parameters the algebra BB_f is semisimple. As before in the case of the q-deformation of Brauer's centralizer algebra, the algebra BB_f is semisimple if q is not a root of unity, and if the weights of the Markov trace are not equal to 0; the latter condition is satisfied if r and r' are not equal to $\pm$ a power of q.

The method used in this paper is similar to the arguments used in [28] for determining the structure of a q-version of Brauer's centralizer algebra, where the existence of a trace functional, derived from a knot polynomial, played a crucial role. The main difference now is that the existence of the Markov trace is derived using quantum groups and the already mentioned formulas for the weights. For this we use properties of quasi-triangular Hopf algebras, mostly due to Drinfeld, and braided tensor categories which are well-known by now (see, e.g., [6,14,17, 27]). One can prove the results in [28] by the same methods, independently of any topological results; this has been done in [17], except for the existence of the 2-variable Markov trace.

We expect that our trace formulas will have further applications. It should be possible to use them, similar as it was done in [28] for the q-Brauer algebra, to construct certain semisimple quotients of the algebras BB_f for q a root of unity and suitable specializations of the other parameters, which are closely related to fusion categories. One of the applications would be the construction of subfactors of the hyperfinite II_1 factor corresponding to half-spin representations of $\mathfrak{so}_N$. We will not carry out this construction in this paper (there already exist other constructions of these subfactors, using more advanced theory of quantum groups). It should be noted, however, that so far not every interesting specialization of parameters of our algebras can be interpreted in terms of known fusion categories.

Our paper is organized as follows. In Section 1 we give background information and notations about Jones' basic construction, Lie algebras $\mathfrak{so}_{2n}$

and $\mathfrak{so}_{2n+1}$, quantum groups, categorical dimension, conditional expectations and Hecke algebras. In Section 2, we recall the decomposition of $V_{m\epsilon} \otimes V^{\otimes f}$ and give a description of $\mathcal{A}_f$ via generators and relations, using R-matrices. In Section 3, we determine uniform formulas for q-dimensions of representations of $U_q\mathfrak{so}_{2n}$, with the number of factors independent of n. In Section 4, we show that $\mathcal{A}_f$ is a quotient of the type B-BMW algebra and that the formulas of Section 3 give the weights of its Markov trace. In the last section we describe the centralizer algebra $\mathcal{A}_f$ in its classical limit $q = 1$, and a corresponding abstract algebra. The construction is analogous to the one of the degenerate affine Hecke algebra, which has been studied before by Drinfeld and Cherednik. If $m = 1$, our limit algebra is a quotient of an algebra defined by Nazarov, and it can be considered a type B analog of Brauer's centralizer algebra.

1. Preliminaries

1.1. Inclusion of algebras

For simplicity, all algebras in this paper are defined over the complex numbers. In particular, a *semisimple* algebra will be a finite direct sum of full matrix rings. $M_n(\mathbb{C})$ denotes the algebra of $n \times n$ matrices over the complex numbers. If A and B are semisimple algebras over $\mathbb{C}$, then $A = \bigoplus A_i$ and $B = \bigoplus B_j$ with $A_i \cong M_{a_i}(\mathbb{C})$ and $B_j \cong M_{b_j}(\mathbb{C})$ for natural numbers a_i and b_j.

Let $A \subset B$ denote the inclusion of algebras, where we always assume that A and B have the same identity. Then any simple B_j module W_j is also an A module. Let g_{ij} be the number of A_i modules in the decomposition of W_j into simple A modules. The matrix $G = (g_{ij})$ is called the inclusion matrix for $A \subset B$.

A Bratteli diagram is a graph with vertices arranged in two lines. This graph describes the inclusion of algebras $A \subset B$. In one line, the vertices are in 1–1 correspondence with the minimal direct summands A_i of A, in the other one with the summands B_j of B. A vertex corresponding to A_i is joined with a vertex B_j by g_{ij} edges.

A *trace* is a linear functional $\mathrm{tr}: B \to \mathbb{C}$ such that $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for all $a, b \in B$. Every trace on $M_n(\mathbb{C})$ is a multiple of the usual trace, i.e. the sum of the diagonal entries. Then any trace on a semisimple algebra B is completely determined by a weight vector $\vec{t} = (t_j)$ where $t_j = \mathrm{tr}(p_j)$ and p_j is a minimal idempotent of B_j. The t_j's are called the weights of the trace. If $\vec{s}$ is the weight vector for the subalgebra A, then $\vec{s} = G\vec{t}$; conversely, any weight vector $\vec{t}$ satisfying $\vec{s} = G\vec{t}$ defines an extension of the trace tr with weight vector $\vec{s}$ on A to B. The annihilator ideal J of tr is defined to be

$$J = \{b \in B\colon\ \mathrm{tr}(ab) = 0 \text{ for all } a \in B\}.$$

A trace on B is called *nondegenerate* if $J = 0$. One can show that a trace is nondegenerate if and only if $t_j \neq 0$ for all j.

One defines a representation π_{tr} of B on B/J where the action is left multiplication. By the trace property

$$\pi_{\mathrm{tr}}(B) \cong B/J.$$

Recall that if tr is nondegenerate there exists an isomorphism $B \stackrel{\cong}{\to} B^*$ (dual of B) defined by $b \to \mathrm{tr}(b\cdot)$, where $\mathrm{tr}(b\cdot)$ is the map $x \to \mathrm{tr}(bx)$. Assuming tr is nondegenerate on A and B and using this isomorphism for A and A^*, one defines a linear map $\varepsilon_A : B \to A$ defined by $\mathrm{tr}(b\cdot)|_A = \mathrm{tr}(\varepsilon_A(b)\cdot)|_A$. The map ε_A is called the trace preserving *conditional expectation*. The element $\varepsilon_A(b) \in A$ is completely determined by

$$\mathrm{tr}\big(\varepsilon_A(b)a\big) = \mathrm{tr}(ba) \quad \text{for all } a \in A.$$

Assume that $A \subset B$ and that B is a subalgebra of C. Furthermore, assume that there exists an element $e \in C$ such that

(i) $e^2 = e$,
(ii) $ebe = e\varepsilon_A(b) = \varepsilon_A(b)e$ for all $b \in B$,
(iii) the map $a \in A \to ae$ is an injective homomorphism with $1e = e$.

An example of this situation is Jones' basic construction [12]. Let B be represented on itself via the left-regular representation. We denote the isomorphic image of B in this representation also by B. When B is regarded as a representation space, it will be denoted B_ξ and its elements b_ξ for $b \in B$. Then C is the set of all linear maps on B_ξ, i.e. $L(B_\xi)$. We define e_A to be the projection onto the subspace $A_\xi \subset B_\xi$ given by $e_A b_\xi = \varepsilon_A(b)_\xi$. One can easily show that e_A satisfies the conditions (i)–(iii) above. The algebra $\langle B, e_A\rangle$ is called Jones' basic construction for $A \subset B$.

Theorem 1.1. *Let A, B, e, e_A,* tr *and ε_A be as above. Then*

(a) *The algebra $\langle B, e_A\rangle$ is equal to the centralizer of the right action ρ of A on B_ξ, given by $\rho(a)b_\xi = ba_\xi$. In particular, it is semisimple.*
(b) *There is a* 1–1 *correspondence between the simple components of A and $\langle B, e_A\rangle$ such that if $p \in A_i$ is a minimal idempotent, pe_A is a minimal idempotent of $\langle B, e_A\rangle$. Under this correspondence, the inclusion matrix for $B \subset \langle B, e_A\rangle$ is the transpose G^{t} of the inclusion matrix for $A \subset B$.*
(c) $\langle B, e_A\rangle = Be_AB$.
(d) *$\langle B, e\rangle$ is a direct sum of full matrix rings which decomposes as*

$$\langle B, e\rangle \cong \langle B, e_A\rangle \oplus \widetilde{B}$$

where $\widetilde{B}$ is an algebra isomorphic to a subalgebra of B. In particular, the ideal generated by e is isomorphic to the semisimple algebra $\langle B, e_A\rangle$.

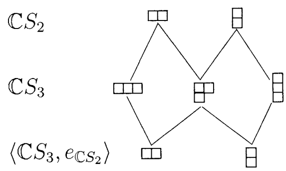


Fig. 1.

Remarks. 1. The theorem above also implies that the map $b \in B \mapsto be$ defines a vector space isomorphism between B and $Be \subset \langle B, e\rangle$. Indeed, if $b \neq 0$, then there exists $b' \in B$ such that $\mathrm{tr}(b'b) \neq 0$. As ε_A is trace preserving, this also means that $\varepsilon_A(b'b) \neq 0$, and hence also $eb'be = \varepsilon_A(b'b)e \neq 0$, by (ii). This implies $be \neq 0$.

2. Statement (b) of this theorem implies that one can compute the structure of $\mathrm{End}_A(B_\xi)$ (and of $\langle B, e_A\rangle$) by reflecting the Bratteli diagram for $A \subset B$ about the line of B.

Example (Fig. 1). Let $\mathbb{C}S_n$ be the group algebra of the symmetric group. $A = \mathbb{C}S_2 \subset B = \mathbb{C}S_3$.

Notice that in this example $\langle \mathbb{C}S_3, e_{\mathbb{C}S_2}\rangle$ has two 3-dimensional irreducible representations.

1.2. *Lie algebras*

1.2.1. *Even orthogonal algebra*

Consider the complex Lie algebra $\mathfrak{so}_{2n}$ with Dynkin diagram given in Fig. 2.

Let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{so}_{2n}$, and let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be an orthonormal basis of $\mathfrak{h}^*$ with respect to the Cartan–Killing form. The simple roots are given by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leqslant i \leqslant n-1$, and $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$. The set of all roots is given by $\{\pm\varepsilon_i \pm \varepsilon_j\colon 1 \leqslant i \neq j \leqslant n\}$. The fundamental weights are $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$ for $1 \leqslant i \leqslant n-2$, $\omega_{n-1} = (\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n)/2$, and $\omega_n = (\varepsilon_1 + \cdots + \varepsilon_n)/2$. We denote by ρ half the sum. of the positive roots. In vector notation, with respect to our chosen orthonormal basis, ρ is given by

$$\rho = (n-1, n-2, \ldots, 0).$$

The set P^+ of the dominant integral weights for $\mathfrak{so}_{2n}$ is given as follows:

$$P^+ = \big\{(\lambda_1, \lambda_2, \ldots, \lambda_n)\colon \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant |\lambda_n| \geqslant 0\big\}$$

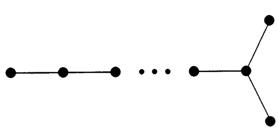


Fig. 2.

where either all λ_i are integers or they are all half-integers. For a given dominant integral weight λ, let V_λ be the irreducible module with highest weight λ. Denote by χ^λ the character of the irreducible module V_λ.

We define the operator q^ρ on an integrable highest weight module M by $q^\rho m = q^{(\rho,\mu)}m$, for a weight vector $m \in M$ with weight μ. Then the q-dimension of V_λ and the q-trace Tr_q are defined by

$$\mathrm{Tr}_q(a) = \mathrm{Tr}\big(q^\rho a\big), \quad a \in \mathrm{End}(V_\lambda), \qquad \dim_q V_\lambda = \mathrm{Tr}_q(1_{V_\lambda}), \tag{1}$$

where Tr is the usual trace on $\mathrm{End}(V_\lambda)$ given by the sum of the diagonal entries. To write down an explicit formula, let

$$[s]_q = \frac{q^s - q^{-s}}{q - q^{-1}}.$$

Then one deduces from Weyl's character formula that

$$\dim_q V_\lambda = \prod_{1 \leqslant i < j \leqslant n} \frac{[2n + \lambda_i + \lambda_j - j - i]_q[\lambda_i - \lambda_j + j - i]_q}{[2n - j - i]_q[j - i]_q}.$$

1.2.2. Odd orthogonal algebra

The complex Lie algebra $\mathfrak{so}_{2n+1}$ has Dynkin diagram as in Fig. 3.

Let $\mathfrak{h}$ denote its Cartan subalgebra and let $\varepsilon_1, \dots, \varepsilon_n$ be an orthonormal basis for $\mathfrak{h}^*$ with respect to the Cartan–Killing form. The simple roots are given by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leqslant i \leqslant n-1$ and $\alpha_n = \varepsilon_n$. The set of roots is given by $\{\pm\varepsilon_i \mp \varepsilon_j, \pm\varepsilon_i \pm \varepsilon_j\colon 1 \leqslant i < j \leqslant n\} \cup \{\pm\varepsilon_i\colon 1 \leqslant i \leqslant n\}$. The fundamental weights are $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$ for $1 \leqslant i \leqslant n$ and $\omega_n = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n)$. The half sum of the positive roots is

$$\rho = (n - 1/2, n - 3/2, \dots, 3/2, 1/2).$$

The set of dominant integral weights is given as follows:

$$P^+ = \big\{(\lambda_1, \dots, \lambda_n)\colon \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0\big\}$$

where all the λ_i's are integers or half-integers simultaneously. From Weyl's character formula we have that

$$\dim_q(V_\lambda) = \prod_{i=1}^{n} \frac{[(2n+1)/2 + \lambda_i - i]_q}{[(2n+1)/2 - i]_q} \times \prod_{1 \leqslant i < j \leqslant n} \frac{[2n + 1 + \lambda_i + \lambda_j - i - j]_q[\lambda_i - \lambda_j + j - i]_q}{[2n + 1 - i - j]_q[j - i]_q}.$$

Fig. 3.

1.3. Quantum groups

We collect certain facts about the Drinfeld–Jimbo deformations of universal enveloping algebras [5,6]; for additional information and details see also [14,17,27].

Let $\mathfrak{g}$ and $\mathfrak{h}$ be as in Section 1.2 and let $A = (a_{ij})_{1\leqslant i,j\leqslant n}$ be the Cartan matrix for $\mathfrak{g}$. Then A is symmetrizable, i.e., there exist nonzero integers d_i, $i = 1, 2, \ldots, n$, such that $d_i a_{ij} = d_j a_{ji}$. We define the quantum group $\boldsymbol{U} = U_q\mathfrak{g}$ as the Hopf algebra over the field $\mathbb{C}(q)$ of rational functions over $\mathbb{C}$ given by generators $X_i^{\pm}$, $K_i^{\pm 1}$ $(1 \leqslant i \leqslant n)$ and relations:

$$K_iK_j = K_jK_i, \qquad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$
$$K_iX_j^{\pm}K_i^{-1} = q^{\pm d_i a_{ij}/2}X_j^{\pm}, \qquad \left[X_i^+, X_j^-\right] = \delta_{ij}\frac{K_i^2 - K_i^{-2}}{q^{d_i} - q^{-d_i}},$$
$$\sum_{s=0}^{1-a_{ij}} \frac{[1-a_{ij}]_{q^{d_i}}!}{[s]_{q^{d_i}}[1-a_{ij}-s]_{q^{d_i}}!}\left(X_i^{\pm}\right)^{1-a_{ij}-s}\left(X_j^{\pm}\right)\left(X_i^{\pm}\right)^s = 0, \quad i \neq j,$$

where

$$[m]_q! = \prod_{j=1}^{m} \frac{q^j - q^{-j}}{q - q^{-1}}.$$

Setting $e_i = X_i^+$, $f_i = X_i^-$ and setting formally $K_i = q^{d_i h_i/2}$, one obtains the relations among the Chevalley generators for the classical universal enveloping algebra in the limit $q \to 1$. $U_q(\mathfrak{g})$ also has a comultiplication $\triangle$ defined by

$$\triangle\left(X_i^{\pm}\right) = K_i \otimes X_i^{\pm} + X_i^{\pm} \otimes K_i^{-1},$$
$$\triangle(K_i) = K_i \otimes K_i.$$

Moreover, the algebra $U_q\mathfrak{g}$ has an antipode γ which, together with the coproduct $\triangle$ makes $U_q\mathfrak{g}$ into a Hopf algebra. Moreover, the quantum group $U_q\mathfrak{g}$ is a quasitriangular Hopf algebra. This means, in particular, that there exists an invertible element $\mathcal{R}$, the universal R-matrix, in a certain completion of $U \otimes U$ which satisfies the relations

$$\mathcal{R}\triangle(a)\mathcal{R}^{-1} = \triangle^{\mathrm{op}}(a) \quad \text{for all } a \in U,$$

where the opposite coproduct is given by $\triangle^{\mathrm{op}} = \sum_a a_{(2)} \otimes a_{(1)}$ if $\triangle(a) = \sum_a a_{(1)} \otimes a_{(2)}$,

$$(\triangle \otimes 1)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad \text{and} \quad (1 \otimes \triangle)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12},$$

where, if $\mathcal{R} = \sum a_i \otimes b_i$ then

$$\mathcal{R}_{12} = \sum a_i \otimes b_i \otimes 1, \qquad \mathcal{R}_{13} = \sum a_i \otimes 1 \otimes b_i,$$
$$\mathcal{R}_{23} = \sum 1 \otimes a_i \otimes b_i.$$

An explicit form of the R-matrix has been obtained in the general case in [15,18], extending work of Drinfeld. The existence of the R-matrix means that for $U_q\mathfrak{g}$-modules V, W, there are natural braiding isomorphisms $\check{R}_{VW}: V\otimes W \to W\otimes V$ given by

$$v\otimes w \to \sum b_i w\otimes a_i v.$$

The braiding axioms (see, e.g., [14, Chapter XIII]) imply in particular the following property: let U, V, W be $U_q(\mathfrak{g})$-modules, then

$$\check{R}_{U\otimes V,W} = \big(\check{R}_{UW}\otimes 1_V\big)\big(1_U\otimes \check{R}_{VW}\big). \tag{2}$$

As one of the consequences of this axiom, one can define, for any $U_q\mathfrak{g}$-module V, a representation of Artin's braid group B_f on f strands in $V^{\otimes f}$ via the map

$$\sigma_i \mapsto \check{R}_i = 1_{i-1}\otimes \check{R}_{VV}\otimes 1_{f-i-1}, \tag{3}$$

where 1_j is the identify map on $V^{\otimes j}$. One can also define a quantum Casimir (see [6, Section 5]; for additional details see also, e.g., [27, XI.7], [17]). More precisely, one derives from $\mathcal{R} = \sum a_i\otimes b_i$ the element $u = \sum \gamma(b_i)a_i$ where γ is the antipode. This element has a well-defined action on each highest weight module of $U_q\mathfrak{g}$. The quantum Casimir C is then defined by $C = q^{\rho}u^{-1}$. Drinfeld then showed the following proposition.

Proposition 1.2. *Let V_λ, V_μ, V_ν be simple $U_q\mathfrak{g}$-modules with highest weights λ, μ, ν, respectively, and such that V_λ is a submodule of $V_\mu\otimes V_\nu$. Then*

$$\big(\check{R}_{V_\mu V_\nu}\check{R}_{V_\nu V_\mu}\big)\big|_{V_\lambda} = q^{c(\lambda)-c(\mu)-c(\nu)}1_{V_\lambda},$$

where for any weight γ the quantity $c(\gamma)$ is given by $(\gamma+2\rho,\gamma)$.

If q is not a root of unity, the representation theory of $U_q\mathfrak{g}$ is similar to the one of the corresponding classical algebra. As in the classical case, each finite dimensional $U_q\mathfrak{g}$-module is a direct sum of its weight spaces, and it is completely reducible. Also, the finite dimensional irreducible modules of $U_q\mathfrak{g}$ are labeled by the dominant integral weights of $\mathfrak{g}$. Moreover, this identification of simple modules also induces an isomorphism between the Grothendieck semirings (i.e. the tensor product rules) of the representations of $\mathfrak{g}$ and $U_q\mathfrak{g}$.

1.4. Categorical dimensions and conditional expectations

We recall the definitions of categorical traces and give some details for a fairly straightforward generalization of the trace operation and conditional expectations. This is based on work by Joyal and Street [13]; for additional details see [14, Chapter XIV] or [27, Chapters I and XL]. All these definitions work for a semisimple rigid ribbon tensor category; however, as we are only interested

Fig. 4.

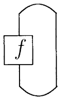
$\mathrm{Tr}_q(f) =$ 

Fig. 5.

in the representation category $\mathcal{C}$ of $U_q\mathfrak{g}$, we will state them for modules over a quasitriangular Hopf algebra $\boldsymbol{U}$, whose axioms are satisfied by $\boldsymbol{U}$ (see [5,6, 14]). In the following, object is synonymous with module over $\boldsymbol{U}$, and morphism between 2 objects V and X means a linear map between V and X which intertwines the $U_q\mathfrak{g}$-action.

We say that an object V in a tensor category $\mathcal{C}$ has a left dual if there exists an object W and morphisms

$$b_V : I \to V \otimes W \quad \text{and} \quad d_V : W \otimes V \to I$$

such that

$$(1_V \otimes d_V)(b_V \otimes 1_V) = 1_V \quad \text{and} \quad (d_V \otimes 1_W)(1_W \otimes b_V) = 1_W.$$

These elements are represented by the pictures in Fig. 4, see also [14, p. 343].

We shall also assume that our category allows a braiding and a twist, i.e. we have R-matrices and a quantum Casimir. We can use this to define the quantum trace on the object V as follows: Let $f : V \to V$ be a morphism. Then the q-trace Tr_q is defined by

$$\mathrm{Tr}_q(f) = d_V \check{R}_{V,W}(C_V f \otimes 1_W) b_V. \tag{4}$$

This can be visualized as in Fig. 5.

If $\dim_q V \neq 0$, the normalized trace tr_q is defined by $(1/\dim_q V)\,\mathrm{Tr}_q$.

Lemma 1.3. *The functional* Tr_q *has the following properties*:

(a) $\mathrm{Tr}_q(fg) = \mathrm{Tr}_q(gf)$ *for all* $f : V \to X$ *and* $g : X \to V$, *and* V, X, *two* $\boldsymbol{U}$-*modules.*
(b) *If* V *is a module of a Drinfeld–Jimbo quantum group,* $\mathrm{Tr}_q(a) = \mathrm{Tr}(q^\rho a)$ *for any morphism* $a : V \to V$, *with* Tr *being the usual trace on* $\mathrm{End}(V)$. *In particular,* Tr_q *does not depend on the choice of* b_V *and* d_V.
(c) *If* V *is a simple* $\boldsymbol{U}$-*module which has a left-dual, then it has nonzero dimension.*

Proof. For a proof of statement (a) see e.g. [27, Chapter I] or [14, Theorem XIV.4.2]. The formula for Tr_q in (b) or Eq. (1) already appears in [6]; see also, e.g., [14, Theorem XIV.6.4], [27, XI] or [17]. Part (c) is also well-known. Here is the argument: Let $\tilde{d}_V = d_V \check{R}_{V,W}(C_V \otimes 1_W)$. Then $\dim_q V = \tilde{d}_V \circ b_V$ and $(b_V \circ \tilde{d}_V)^2 = (\dim_q V) b_V \circ \tilde{d}_V$. By irreducibility of V and Frobenius duality, $\dim(\mathrm{Hom}(I, V \otimes W)) = \dim(\mathrm{End}(V)) = 1$. Hence $\dim_q V = 0$ would imply $b_V \circ \tilde{d}_V = 0$. But then also

$$0 = (1_V \otimes d_V)(b_V \otimes 1_V)\big(\tilde{d}_V \otimes 1_V\big) = \big(\tilde{d}_V \otimes 1_V\big),$$

by the axioms of left-duality. As $\check{R}_{V,W}(C_V \otimes 1_V)$ is invertible, this would imply $d_V = 0$, contradicting the duality axiom. □

In the following we will always assume that V has a left-dual, which we denote by W. Conditional expectations can also be very naturally defined using our categorical definitions. Let X be an object. Let $A = \mathrm{End}(X) \cong A \otimes 1_V \subset B = \mathrm{End}(X \otimes V)$. We define the map ε_A from B onto A by

$$\varepsilon_A(b) = \frac{1}{\dim_q V}\big(1_X \otimes d_V \check{R}_{VW}\big)(1_X \otimes C_V \otimes 1_W)(b \otimes 1_W)1_X \otimes b_V;$$

in the tangle picture, $\varepsilon_A(b)$ is obtained from b by closing up the tangle with color V and renormalizing by $1/\dim_q V$, see Fig. 6.

We also define the element $e_V = b_V d_V \check{R}_{VW}(C_V \otimes 1_W)/\dim_q V$.

Proposition 1.4. (a) *The map ε_A is the trace preserving conditional expectation from B to A with respect to the normalized quantum trace* tr_q.

(b) *Let $X = X_1 \otimes X_2$, and let $A_i = \mathrm{End}(X_i)$, $i = 1, 2$. If $b \in \mathrm{End}(X)$ is of the form $b_1 \otimes b_2$, then $\varepsilon_A(b) = b_1 \otimes \varepsilon_{A_2}(b_2)$, where ε_{A_2} is the conditional expectation from* $\mathrm{End}(X_2 \otimes V)$ *onto A_2.*

(c) *If $X = V' \otimes V$ for objects V, V', and $b = 1_{V'} \otimes \check{R}_{VV}$, then $\varepsilon_A(b) =$* $\mathrm{tr}_q(\check{R}_{VV})1_X$.

(d) *The algebras A, B and the element e_V satisfy the conditions of Theorem* 1.1.

Proof. (a) It suffices to show that $\mathrm{tr}_q(a\varepsilon(b)) = \mathrm{tr}_q(ab)$. We do this using graphical calculus. Replacing $f \otimes g$ in the picture on p. 356 in [14] by $(a \otimes 1) \circ b$, and applying the same operations as on that page until the end of the next to last

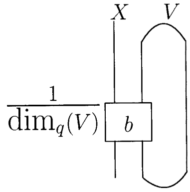
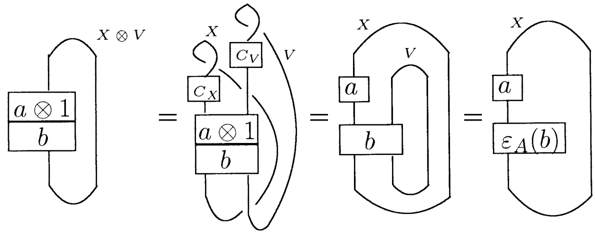


Fig. 6. Conditional expectation.

row, we get the first equality of the pictures below. The second equality is obvious. It now remains to observe that the last picture is equal to $\mathrm{tr}_q(a\varepsilon_A(b))$. The more algebraic minded reader may check that the changes in the pictures correspond to applying algebraic identities coming from the braiding axioms:

$$\mathrm{Tr}_q\big((a\otimes 1)b\big) =$$



$$= \mathrm{Tr}_q\big(a\varepsilon_A(b)\big).$$

Statement (b) is most easily proved by pictures, using [14, Theorem XIV.4.2(b)] and its proof again. Part (c) follows from this. To prove statement (d), it suffices to show that e_V satisfies the conditions (i)–(iii) of Theorem 1.1. It follows from the definition of Tr_q, Eq. (4), that $\dim_q V = \mathrm{Tr}_q(1_V) = d_V\check{R}_{VW}(C_V\otimes 1_V)b_V$. From this one easily deduces that $e_V^2 = e_V$. For condition (ii), observe that

$$\begin{aligned}
&(1_X\otimes e_V)(b\otimes 1_W)(1_X\otimes e_V)\\
&\quad=\frac{1}{(\dim_q V)^2}\big(1_X\otimes b_V d_V\check{R}_{VW}(C_V\otimes 1)\big)(b\otimes 1_W)\\
&\qquad\times\big(1_X\otimes b_V d_V\check{R}_{VW}(C_V\otimes 1_W)\big)\\
&\quad=\frac{1}{(\dim_q V)^2}(1_X\otimes b_V)\big(1_X\otimes d_V\check{R}_{VW}(C_V\otimes 1)\big)(b\otimes 1_W)(1_X\otimes b_V)\\
&\qquad\times\big(1_X\otimes d_V\check{R}_{VW}(C_V\otimes 1_W)\big)\\
&\quad=\frac{1}{(\dim_q V)^2}(1_X\otimes b_V)\big(\varepsilon_A(b)\otimes 1_V\otimes 1_W\big)\big(1_X\otimes d_V R_{VW}(C_V\otimes 1_W)\big)\\
&\quad=\frac{1}{\dim_q V}\varepsilon_A(b)\otimes e_V.
\end{aligned}$$

Finally, for condition (iii), observe that the image of e_V is isomorphic to I and $X\otimes I\cong X$. Hence $e_V\,\mathrm{End}(X\otimes V\otimes W)e_V\cong \mathrm{End}(X) = A$. □

We shall also assume that the modules $X\otimes V\otimes W$ as well as X are semisimple. Then we can write

$$X\otimes V\otimes W = (X\otimes V\otimes W)_{\mathrm{old}}\oplus (X\otimes V\otimes W)_{\mathrm{new}},$$

where $(X\otimes V\otimes W)_{\mathrm{new}}$ is the maximum submodule M of $X\otimes V\otimes W$ such that $\mathrm{Hom}(X, M) = 0$.

Proposition 1.5. *The algebra* $\mathrm{End}((X\otimes V\otimes W)_{\mathrm{old}})$ *is isomorphic to* Be_VB.

Proof. If Z is a simple submodule of the new part, and $b \in B$, then $e_V b(Z)$ would be isomorphic to Z or zero. We can dismiss the first case, as the image of e_V only contains 'old' modules. Hence $Be_V B$ acts as zero on $(X \otimes V \otimes W)_{\text{new}}$. Similarly, one shows that $Be_V B$ maps $(X \otimes V \otimes W)_{\text{old}}$ into itself. Hence $Be_V B \subset \text{End}((X \otimes V \otimes W)_{\text{old}})$.

Let Y be any object. Then it is well-known that $\text{Hom}(Y, V \otimes X) \cong \text{Hom}(W \otimes Y, X)$. We give here the argument in our setting, for the reader's convenience. We define for $f \in \text{Hom}(Y, V \otimes X)$ and $g \in \text{Hom}(W \otimes Y, X)$ the morphisms

$$\alpha(f) = (d_V \otimes 1)(1 \otimes f) \quad \text{and} \quad \beta(g) = (1 \otimes g)(b_V \otimes 1).$$

Then $\alpha(\beta(g)) = g$ and $\beta(\alpha(f)) = f$ for all $f \in \text{Hom}(Y, V \otimes X)$ and $g \in \text{Hom}(W \otimes Y, X)$, which can probably most easily be checked using graphical calculus. This shows that $\dim(\text{Hom}(Y, V \otimes X)) = \dim(\text{Hom}(W \otimes Y, X))$. Using the braiding isomorphisms, we also get equality of the dimensions of Hom spaces if we permute the tensor factors in one or both expressions in the last sentence.

Because of semisimplicity X can be decomposed as a direct sum of simple objects in the form $X = \bigoplus_\mu a_\mu V_\mu$. This implies that $\text{End}(X) = \sum_\mu M_{a_\mu}$, where M_d is a full $d \times d$ matrix ring. Moreover, let $g_{\lambda\mu}$ be the multiplicity of the simple object V_λ in $V_\mu \otimes V$. Then it follows from the previous paragraph that $g_{\lambda\mu}$ is also the multiplicity of V_μ in $V_\lambda \otimes W$. Hence the multiplicity of the simple object V_μ in $X \otimes V \otimes W$ is equal to $\sum_{\nu,\lambda} g_{\lambda\mu} g_{\lambda\nu} a_\nu$. Hence the dimension vector of $\text{End}((X \otimes V \otimes W)_{\text{old}})$ is equal to $G^{\mathrm{t}} G\vec{a}$, i.e. the inclusion matrix for $B \subset \text{End}((X \otimes V \otimes W)_{\text{old}})$ is given by G^{t}. As this is also the inclusion matrix for $B \subset Be_V B$ (see Theorem 1.1(b)), the claimed isomorphism follows. □

1.5. *Hecke algebras*

An algebra that will play an important role in this paper is the Hecke algebra of type B. We now define this algebra and give some basic facts about this algebra.

Definition 1.6. The Hecke algebra of type B_n, denoted by $HB_n(q, r')$ is the free complex algebra with generators t, $g_1, \ldots, g_{n-1}$ and parameters $q, r' \in \mathbb{C}$ satisfying relations:

(1) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad i = 1, \ldots, n-2;$

(2) $g_i g_j = g_j g_i, \quad |i-j| > 1;$

(3) $g_i^2 = (q - q^{-1}) g_i + 1;$

(4) $t^2 = (r'+1)t - r';$

(5) $t g_1 t g_1 = g_1 t g_1 t;$

(6) $t g_i = g_i t, \quad i \geqslant 2.$

We remark that the definition given in this paper is slightly different to the one found in the literature. The difference is in relation (4), where we take the eigenvalues of t to be r' and 1 as opposed to r' and -1.

Hoefsmit [9] showed that this algebra is semisimple and that the irreducible representations are indexed by pairs of partitions (or Young diagrams) of n. It is well know that if $r' = -1$ and $q = 1$ then $H_n(1, -1)$ is isomorphic to the group algebra of the hyperoctahedral group.

The irreducible matrix representations of $HB_n(q, r')$ are defined on a vector space $V_{(\alpha,\beta)}$ with basis elements indexed by standard Young tableaux of shape (α, β). The matrix representations of the generator g_i can be constructed from 2×2 blocks of the form

$$M(d, r') = \frac{1}{1 - q^d r'} \begin{pmatrix} q-1 & 1 - q^{d+1} r' \\ q(1 - q^{d-1} r') & -q^d r'(q-1) \end{pmatrix}, \tag{5}$$

where $d = c(i) - c(i-1)$ is known as the "axial distance" of the numbers i and $i-1$ in a tableau, and where the contents $c(i)$ is the difference of the row index minus the column index of the box containing i. For details of this construction see [10] or [1].

A subalgebra that will be of interest in the last section when we discuss the classical limit is the following. Set $t_1 = t$ and define inductively the elements $t_{i+1} = g_i t_i g_i$ $(1 \leqslant i \leqslant n-1)$. The elements $t_1, t_2, \ldots, t_n$ generate an abelian subalgebra of $H_n(q, r')$. For a proof of this result see [1, Lemma 3.3].

2. Centralizer algebras

2.1. Tensor product rules for orthogonal Lie algebras

In the following we describe the decomposition of the tensor product of an irreducible $\mathfrak{so}_N$ module V_λ with the vector representation V of $\mathfrak{so}_N$. These rules are well-known. We shall treat the even- and odd-dimensional cases separately.

Let V be the fundamental $U_q(\mathfrak{so}_{2n})$-module, i.e. the analog of the $2n$-dimensional vector representation of $SO(2n)$, and let V_λ be a simple $U_q(\mathfrak{so}_{2n})$-module with highest weight λ. Then the decomposition of the tensor product $V_\lambda \otimes V$ is given by

$$V_\lambda \otimes V \cong \bigoplus_{\mu \leftrightarrow \lambda} V_\mu, \tag{6}$$

where the sum is over all dominant weights μ of the form $\mu = \lambda \pm \varepsilon_i$.

Let $\epsilon = (1/2, \ldots, 1/2)$, which is the highest weight of one of the spinor representations V_ϵ, and let m be a positive integer. Then we obtain as a special case of the formula above:

$$V_{m\epsilon} \otimes V \cong V_{(m/2+1, m/2, \ldots, m/2)} \oplus V_{(m/2, \ldots, m/2, m/2-1)}. \tag{7}$$

We will be interested in the decomposition of tensor products of the form $V_{m\epsilon} \otimes V^{\otimes f}$. Let $\mathcal{A}_f = \mathrm{End}_{U_q(\mathfrak{so}_{2n})}(V_{m\epsilon} \otimes V^{\otimes f})$. As $U_q(\mathfrak{so}_{2n})$ is semisimple for q not a root of unity, the simple components of $\mathcal{A}_f$ are labeled by the isomorphism classes of simple representations appearing in $(V_{m\epsilon} \otimes V^{\otimes f})$, and the dimension of a simple module, $W_{(\mu,f)}$, will be equal to the multiplicity of the simple $U_q(\mathfrak{so}_{2n})$-module V_μ in that tensor power.

For given $m \in \mathbb{N}$, $m > 0$, we define the labeling set $\Lambda_f(2n,m)$ as the set of all highest weights λ for which V_λ appears in $V_{m\epsilon} \otimes V^{\otimes f}$. A path t of length f is a sequence t of dominant weights

$$t\colon \bigl(\lambda^{(0)} = m\epsilon,\ \lambda^{(1)}, \dots, \lambda^{(f)}\bigr),$$

such that $V_{\lambda^{(i+1)}} \subset V_{\lambda^{(i)}} \otimes V$. This means, in particular, that $\lambda^{(i+1)} - \lambda^{(i)} = \pm\varepsilon_j$ for some $j = 1, 2, \dots, n$. We shall also use the notation $t(i) = \lambda^{(i)}$, and refer to $t(i-1)$ and $t(i+1)$ as predecessor and successor of $t(i)$, respectively. We denote the set of all paths of length f by $\mathcal{P}_f(2n,m)$.

Lemma 2.1. (a) *The elements of $\Lambda_f(2n,m)$ are of the form $\lambda = m\epsilon + \mu$ where $\mu \in \mathbb{Z}^n$ such that λ is a dominant weight of $\mathfrak{so}_{2n}$ and such that $f - \sum |\mu_i|$ is even and nonnegative.*

(b) *Let $f(\lambda)$ be the smallest number f such that $V_{m\epsilon} \otimes V^{\otimes f}$ has a submodule isomorphic to V_λ. Then $f(\lambda) = \|\lambda - m\epsilon\|_1$.*

(c) *If $f(\gamma) - f(\lambda) = 2$, then there exist at most 2 weights μ with $f(\mu) - f(\lambda) = 1$ which can occur in a path between λ and γ.*

(d) $\mathrm{Hom}(V_{m\epsilon} \otimes V^{\otimes f}, V_{m\epsilon} \otimes V^{\otimes f+1}) = 0$.

(e) *Let $W_{(\lambda,f)}$ denote a simple $\mathcal{A}_f$-module. Then its dimension can be computed inductively by the restriction rule*

$$W_{(\lambda,f)} \cong \bigoplus_{(\mu,f-1)} W_{(\mu,f-1)},$$

where the summation goes over all $\mu \in \Lambda_{f-1}(2n,m)$ for which $\lambda - \mu = \omega$ is a weight of V. In particular, the dimension of $W_{(\lambda,f)}$ is equal to the number of paths in $\mathcal{P}_f(2n,m)$ which end in λ.

Proof. Recall the description of the set P^+ of dominant integral weights of $\mathfrak{so}_{2n}$ from Section 1.2. The restriction rule in (e) follows by induction using the branching rule. As a weight ω of V is of the form $\omega = \pm\varepsilon_j$, statement (a) follows from (e) by induction. Statement (d) now is an obvious consequence of (a). Similarly, statement (b) follows from the branching rules by induction. To prove (c), observe that $\gamma - \lambda = \pm\varepsilon_{i_1} \pm \varepsilon_{i_2}$, and $\mu - \lambda = \pm\varepsilon_k$. As $\gamma - \mu = (\gamma - \lambda) - (\mu - \lambda)$, we get $1 = \|\gamma - \mu\|_1 = \|\pm\varepsilon_{i_1} \pm \varepsilon_{i_2} - (\pm\varepsilon_k)\|$. It is now easy to see that this is possible only if $k = i_j$ for $j = 1$ or 2, and with ε_{i_j} having the same sign as ε_k. This proves (c). $\square$

Let now V be the fundamental $U_q(\mathfrak{so}_{2n+1})$-module, i.e. the analog of the $(2n+1)$-dimensional vector representation of $\mathfrak{so}_{2n+1}$, and let V_λ be any simple $U_q(\mathfrak{so}_{2n+1})$-module with highest weight λ. Then we have

$$V_\lambda \otimes V \cong \bigoplus_{\mu \leftrightarrow \lambda} V_\mu$$

where the sum is over all dominant weights μ of the form $\mu \pm \varepsilon_i$ or $\mu = \lambda$ if $\lambda_n > 0$; if $\lambda_n = 0$, we have to leave out $\mu = \lambda$. Hence if $\epsilon = (1/2, \ldots, 1/2)$, the highest weight of the spin representation, V_ϵ, then we have

$$V_\epsilon \otimes V \cong V_\epsilon \otimes V_{(1,0,\ldots,0)+\epsilon}. \tag{8}$$

We will be interested in the decomposition $V_\epsilon \otimes V^{\otimes f}$. Define the labeling set $\Gamma(n, f)$ as the set of all highest weights μ such that V_μ is in the decomposition of $V_\epsilon \otimes V^{\otimes f}$. The diagram in Fig. 7 describes this decomposition. One reads off easily that the vertices in the fth line (starting at line 0) are labeled by the Young diagrams with at most f boxes.

Lemma 2.2. *Let λ be a dominant weight of $\mathfrak{so}_N$ and let V_{μ_1}, V_{μ_2} be irreducible submodules of $V_\lambda \otimes V$, with $\mu_1 \neq \mu_2$. Then $c(\mu_1) \neq c(\mu_2)$ except if N is even, $\lambda_n = 0$ and $\mu_i - \lambda = \pm\varepsilon_n$ for $i = 1, 2$.*

Proof. Let ω be a weight of V. Recall that $c(\gamma) = (\gamma + 2\rho, \gamma)$ for any dominant weight γ. Then it is easy to check that

$$c(\lambda + \omega) = c(\lambda) + 2(\lambda + \rho, \omega) + (\omega, \omega).$$

Hence, if $\mu_i = \lambda + \eta_i$ and η_i is a weight of V, $i = 1, 2$, we obtain

$$c(\mu_1) - c(\mu_2) = 2(\lambda + \rho, \eta_1 - \eta_2).$$

If $N = 2n$ is even, any weight of V is of the form $\pm\varepsilon_j$. It is easy to check that $\eta_1 - \eta_2$ is either a root of $\mathfrak{so}_{2n}$ or it is equal to $\pm 2\varepsilon_j$ for some j, $1 \leqslant j \leqslant n$. As $\lambda + \rho$ is in the dominant Weyl chamber, $(\lambda + \rho, \alpha) \neq 0$ for any root α, and $(\lambda + \rho, \varepsilon_j) = 0$ only if $j = n$ and $\lambda_n = 0$.

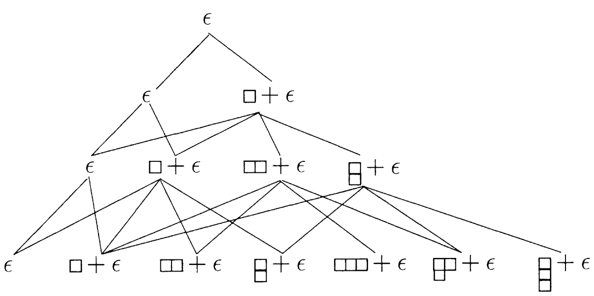


Fig. 7.

If $N = 2n+1$ is odd, one checks similarly that $\eta_1 - \eta_2$ is either a root of $\mathfrak{so}_{n+1}$, or it is equal to $\pm\varepsilon_j$ or $\pm 2\varepsilon_j$ for some j, $1 \leqslant j \leqslant n$. As $2\rho_n = 1$, one deduces that $c(\mu_1) \neq c(\mu_2)$. $\square$

2.2. *Algebraic description of $\mathcal{A}_f$*

Let V and $V_{m\epsilon}$ be as in the previous section, with $m = 1$ if N is odd. We define elements of $\mathrm{End}_U(V_{m\epsilon} \otimes V^{\otimes f})$ by

$$\check{R}_i = 1_{V_{m\epsilon}} \otimes 1_{i-1} \otimes \check{R}_{VV} \otimes 1_{f-i-1} \quad \text{and}$$
$$T = q^{N-2+m} \check{R}_{V V_{m\epsilon}} \check{R}_{V_{m\epsilon} V} \otimes 1_{f-1}.$$

We also define the matrices E_i for $i = 1, \ldots, f-1$ by the following equation:

$$\check{R}_i - \check{R}_i^{-1} = \left(q - q^{-1}\right)(1 - E_i). \tag{9}$$

If we define the element $E \in \mathrm{End}(V^{\otimes 2})$ by a similar equation from the element $\check{R}$, one checks easily that it is a multiple of the projection onto the trivial representation $\subset V^{\otimes 2}$. The following lemma is an easy consequence of Drinfeld's results on the quantum Casimir [6], and part (a) has explicitly appeared before, e.g., in [17,24,28].

Lemma 2.3. (a) *The matrices $\check{R}_i$ have eigenvalues q, $-q^{-1}$ and q^{1-N}.*
(b) *The matrix T has eigenvalues q^{N+2m-2} and* 1.

Proof. It follows from Eq. (7) and Proposition 1.2 that the element $\check{R}_{V V_{m\epsilon}} \check{R}_{V_{m\epsilon} V}$ has eigenvalues q^m and q^{2-N-m}, and that the element $\check{R}_{VV}^2$ has eigenvalues $q^{\pm 2}$ and q^{2-4n}; from the last statement one can easily deduce that $\check{R}_{VV}$ has eigenvalues q, $-q^{-1}$, and q^{1-N} (see, e.g., [17] for more details). $\square$

Proposition 2.4. *The transformations T, $\check{R}_i$ satisfy the following relations*:

(1) $\check{R}_i \check{R}_j = \check{R}_j \check{R}_i, \quad |i-j| > 1;$

(2) $\check{R}_i \check{R}_{i+1} \check{R}_i = \check{R}_{i+1} \check{R}_i \check{R}_{i+1}, \quad 1 \leqslant i \leqslant f-2;$

(3) $\check{R}_i T = T \check{R}_i, \quad i > 1;$

(4) $\check{R}_1 T \check{R}_1 T = T \check{R}_1 T \check{R}_1;$

(5) $\left(\check{R}_i - q^{1-N}\right)(\check{R}_i - q)\left(\check{R}_i + q^{-1}\right) = 0, \quad 1 \leqslant i \leqslant f-1;$

(6) $E_i \check{R}_{i-1}^{\pm 1} E_i = q^{\pm(N-1)} E_i \quad \textit{and} \quad E_i \check{R}_{i+1}^{\pm 1} E_i = q^{\pm(N-1)} E_i;$

(7) $T^2 = \left(q^{2m+N-2} + 1\right) T - q^{2m+N-2};$

(8) $E_1 T E_1 = a E_1, \quad \textit{where } a = -\dfrac{(q^{N+2m-2}+1)(q^N - 1)}{(q^2-1)};$

(9) $E_i \check{R}_i = q^{1-N} E_i.$

Proof. Relations (1)–(4) are consequences of the braiding properties of the R-matrices (see, e.g., [14]). Relations (5) and (7) follow from Lemma 2.3 (it also follows from Drinfeld's quantum Casimir approach that $\check{R}_i$ and T are diagonalizable for generic q). It follows from (5) and the definition of E by the equation $\check{R} - \check{R}^{-1} = (q - q^{-1})(1 - E)$ that E is a multiple of the projection onto the trivial module $\subset V^{\otimes 2}$. This implies (9). One checks that $E_i^2 = (\dim_q V)E_i$, where $\dim_q V = (q^{N-1} - q^{1-N})/(q - q^{-1}) + 1$. Hence $E_i = b_V \circ \tilde{d}_V$ (see proof of Lemma 1.3). In particular, $E_1 T E_1 = (\dim_q V)\varepsilon(T)E_1 = (\dim_q V)\,\mathrm{tr}_q(T)E_1$. As the eigenvalues of T are known as well as the values of the traces of its eigenprojections, via the formulas for the q-dimensions, relation (8) follows from a straightforward computation. Relation (6) is checked similarly (or see [17, Proposition 5.10]). □

As $E(V^{\otimes 2})$ is isomorphic to the trivial representation, the image of E_f is isomorphic to $V_{m\epsilon} \otimes V^{\otimes f-2}$. This implies that the dominant weights indexing the irreducible representations in this image are "old" weights. Thus, $E_f|_{(V_{m\epsilon}\otimes V^{\otimes f})_{\mathrm{new}}} = 0$. This implies that Eq. (9) becomes

$$\big(\check{R}_i - \check{R}_i^{-1}\big)\big|_{(V_{m\epsilon}\otimes V^{\otimes f})_{\mathrm{new}}} = \big(q - q^{-1}\big)\big|_{(V_{m\epsilon}\otimes V^{\otimes f})_{\mathrm{new}}}. \tag{10}$$

Let $\widetilde{\mathcal{A}}_f$ be the algebra generated by T and $\check{R}_i$, $i = 1, 2, \ldots, f-1$, and let $(\mathcal{A}_f)_{\mathrm{new}}$, respectively $(\widetilde{\mathcal{A}}_f)_{\mathrm{new}}$, be the restrictions of the algebras $\mathcal{A}_f$, respectively $(V_{m\epsilon} \otimes V^{\otimes f})_{\mathrm{new}}$. It is easy to check from the last equation that $(\widetilde{\mathcal{A}}_f)_{\mathrm{new}}$ is a quotient of the Hecke algebra HB_f of type B_f. Similarly, we define by $\mathcal{P}_f(N,m)_{\mathrm{new}}$ the subset of $\mathcal{P}_f(N,m)$ consisting of all paths $t = (\lambda^{(i)})$ for which $V_{\lambda^{(i)}}$ is in $(V_{m\epsilon} \otimes V^{\otimes i})_{\mathrm{new}}$ for $i = 0, 1, \ldots, f$.

To show that $(\widetilde{\mathcal{A}}_f)_{\mathrm{new}}$ coincides with $(\mathcal{A}_f)_{\mathrm{new}}$, we need different representations of these algebras. These are obtained by using a q-version of the a generalization of the so-called Jucys–Murphy approach; this generalization can essentially be found in [17,23]. Basically, it is a way to rederive representations of the Hecke algebras of type B, which were already found by Hoefsmit [10]. As we need the same arguments for certain specializations of these Hecke algebras which are not explicitly covered in the previously mentioned papers, we will review the construction here:

We define elements $M_f \in \mathcal{A}_f$ inductively by

$$M_1 = T \quad \text{and} \quad M_{f+1} = \check{R}_f (M_f \otimes 1)\check{R}_f.$$

It follows from the braiding axioms that $M_{f+1} = \check{R}_{V, V_{m\epsilon}\otimes V^{\otimes f}} \check{R}_{V_{m\epsilon}\otimes V^{\otimes f}, V}$. In particular, if $W \subset V_{m\epsilon} \otimes V^{\otimes f}$ is a simple submodule, then $(M_{f+1})|_{W\otimes V} = \check{R}_{VW}\check{R}_{WV}$. Moreover, the eigenvalues of $(M_{f+1})|_{W\otimes V}$ can be computed using the tensor product decomposition of $W \otimes V$ and the formula in Proposition 1.2. More precisely, let $c(W)$ and $c(V)$ denote the values by which the Casimir acts on W and V, respectively. If $W \otimes V = \bigoplus_\mu V_\mu$, and if p_γ is the projection onto the module V_γ with kernel the remaining summands, we obtain

$$(M_{f+1})|_{W\otimes V} = \sum_\mu q^{c(\mu)-c(W)-c(V)} p_\mu. \tag{11}$$

Lemma 2.5. *For each $t \in \mathcal{P}_f(N,m)_{\mathrm{new}}$ there exists an idempotent $0 \neq p_t \in \widetilde{\mathcal{A}}_f$, such that $p_t p_{\tilde{t}} = 0$ for $t \neq \tilde{t}$, and such that the image of p_t is an irreducible $U_q\mathfrak{g}$-module whose highest weight is labeled by the endpoint of t. Moreover, the image of $\sum_t p_t$ is equal to $(V_{m\epsilon} \otimes V^{\otimes f})_{\mathrm{new}}$, where the summation goes over the paths t in $\mathcal{P}_f(N,m)_{\mathrm{new}}$.*

Proof. We prove the claim by induction on f, with $f = 1$ being trivially true. Now let $t \in \mathcal{P}_{f+1}(N,m)_{\mathrm{new}}$, and let t' be the path in $\mathcal{P}_f(N,m)_{\mathrm{new}}$ obtained by removing $\lambda^{(f+1)}$ from t. By induction assumption, $p_{t'} \in (\widetilde{\mathcal{A}}_f)_{\mathrm{new}}$ exists, and its image W is irreducible with highest weight $\lambda^{(f)}$. By definition $M_{f+1} \in \mathcal{A}_{f+1}$, in particular the operator $\check{R}_{VW}\check{R}_{WV}$ is in $\widetilde{\mathcal{A}}_{f+1}$. By Proposition 1.2 and Lemma 2.2, it has as many distinct eigenvalues as $W \otimes V$ has irreducible components, except possibly if N and m are even and $\lambda_N^{(f)} = 0$. Except for this special case, the eigenprojections of $\check{R}_{VW}\check{R}_{WV}$ coincide with the projections onto the irreducible submodules in $W \otimes V$. We obtain from this the definition of p_t as a subprojection of $p_{t'}$.

In the latter case, the Casimirs would only coincide for the irreducible components of $W \otimes V$ with highest weights $\lambda^{(f)} \pm \epsilon_N$. However, one checks easily, using Lemma 2.1(b), that in this case only one of the 2 weights can be new. Hence we can use the same argument as before for $(\widetilde{\mathcal{A}}_f)_{\mathrm{new}}$. □

Let $W_{(\lambda,f)}$ be a simple $\mathcal{A}_f$-module. By Lemma 2.1(f) and Lemma 2.5, we can find a basis (v_t) for $W_{(\lambda,f)}$ which is labeled by the paths t in $\mathcal{P}_f(N,m)$ with endpoint λ, and such that v_t spans the image of p_t in $W_{(\lambda,f)}$. It is easy to check that $W_{(\lambda,f)}$ is a simple $(\mathcal{A}_f)_{\mathrm{new}}$-module only if $f(\lambda) = f$ (see Lemma 2.1(b)). Moreover, if t: $(\lambda^{(j)})$ is a basis path for $W_{(\lambda,f)}$, we also have $f(\lambda^{(j)}) = j$ (i.e. $V_{\lambda^{(j)}}$ appears for the first time in $V_{m\epsilon} \otimes V^{\otimes j}$). Then one easily checks that $\check{R}_i v_t$ is a linear combination of v_t and at most one other basis vector, say v_s; here s coincides with t except for $\lambda^{(i)}$ (see Lemma 2.1(c)). The case with 1 path is trivial, so we assume we have two paths. It follows from Eq. (11) and the proof of the last lemma that

$$M_i p_t = q^{c(\lambda^{(i+1)})-c(\lambda^{(i)})-c(V)} p_t. \tag{12}$$

By definition of Murphy element we have $M_i = \check{R}_i(M_{i-1} \otimes 1)\check{R}_i$, which implies $\check{R}_i^{-1} = M_i^{-1}\check{R}_i(M_{i-1} \otimes 1)$. M_i and $M_{i-1} \otimes 1$ act diagonally on the path basis, with the entries given by Eq. (12). Since $\check{R}_i - \check{R}_i^{-1} = (q - q^{-1})1$,

$$\check{R}_i - M_i^{-1}\check{R}_i(M_{i-1}\otimes 1) = \big(q - q^{-1}\big)1.$$

Then the diagonal entries of $\check{R}_i$ restricted to the paths s and t are given by

$$\big(\check{R}_i\big)_{jj} = \frac{q - q^{-1}}{1 - q^{2c(\lambda_j^{(i)}) - c(\lambda^{(i+1)}) - c(\lambda^{(i-1)})}}, \quad j \in \{s, t\}.$$

where $\lambda_j^{(i)}$ is the ith diagram in the path j. It follows from Lemma 2.2 that the two diagonal entries $(\check{R}_i)_{jj}$, $j \in \{s, t\}$ are distinct. Hence $\check{R}_i$ acts on $\operatorname{span}\{v_s, v_t\}$ as a matrix with eigenvalues q and $-q^{-1}$. It follows that $(\check{R}_i)_{ss}(\check{R}_i)_{tt} - (\check{R}_i)_{st}(\check{R}_i)_{ts} = -1$. Clearly $(\check{R}_f)_{ss}(\check{R}_f)_{tt} \neq -1$. Thus the off-diagonal entries are nonzero.

Proposition 2.6. *The algebra $(\mathcal{A}_f)_{\mathrm{new}} = \operatorname{End}_{U_q\mathfrak{g}}((V_{m\epsilon}\otimes V^{\otimes f})_{\mathrm{new}})$ is generated by the restrictions of T and $\check{R}_i$, $i = 1, 2, \ldots, f-1$ to $(V_{m\epsilon}\otimes V^{\otimes f})_{\mathrm{new}}$.*

Proof. The claim is proved by induction on f for N even. The case with N odd can be done similarly. For $f = 1$, $V_{m\epsilon}\otimes V \cong (V_{m\epsilon}\otimes V)_{\mathrm{new}}$. It decomposes into the direct sum of two nonequivalent irreducible representations, see Eq. (7). On the other hand, T has two distinct eigenvalues, hence must generate an algebra of dimension at least 2.

Assume the proposition is true for $f-1 \in \mathbb{N}$. Let $W_{(\lambda,f)}$ be a simple $(\mathcal{A}_f)_{\mathrm{new}}$-module. We want to show that it is also a simple $\widetilde{\mathcal{A}}_f$-module. It follows from Lemma 2.1(e) that

$$W_{(\lambda,f)} \cong \bigoplus_{(\mu,f-1)} W_{(\mu,f-1)}$$

as an $\mathcal{A}_{f-1}$-module, where the summation goes over all dominant weights μ with $f(\mu) = f-1$ for which $\lambda - \mu$ is a weight in V. By induction assumption, all modules $W_{(\mu,f-1)}$ are simple $\widetilde{\mathcal{A}}_{f-1}$-modules. One checks from the tensor product rules that for any two labels μ, $\tilde{\mu}$ of the sum above, we can find paths s, t ending in λ such that $s(i) = t(i)$ for all $i \neq f-1$, and $s(f-1) = \mu$, $t(f-1) = \tilde{\mu}$. Hence $v_s \in W_{(\mu,f-1)}$ and $v_t \in W_{(\tilde{\mu},f-1)}$. It follows from the discussion before this proposition that $\check{R}_{f-1}$ acts on $\operatorname{span}\{v_s, v_t\}$ by a matrix with nonzero off-diagonal entries. Hence $W_{(\mu,f-1)}$ and $W_{(\tilde{\mu},f-1)}$ are in the same $\widetilde{\mathcal{A}}_f$-submodule of $W_{(\lambda,f)}$. As μ, $\tilde{\mu}$ were arbitrary, $W_{(\lambda,f)}$ is an irreducible $\widetilde{\mathcal{A}}_f$-module.

The equality $(\widetilde{\mathcal{A}}_f)_{\mathrm{new}} = (\mathcal{A}_f)_{\mathrm{new}}$ will follow as soon as we have shown that the $\mathcal{A}_f$-modules $W_{(\gamma,f)}$ and $W_{(\delta,f)}$ are also nonisomorphic as $\widetilde{\mathcal{A}}_f$-modules, for $\gamma \neq \delta$, with $f(\delta) = f = f(\gamma)$. This is easy to check directly for $f = 0$ and $f = 1$. For $f > 1$ one can check that $W_{(\gamma,f)}$ differs from $W_{(\delta,f)}$ already as an $\widetilde{\mathcal{A}}_{f-1}$-module by finding an $\widetilde{\mathcal{A}}_{f-1}$-submodule of $W_{(\gamma,f)}$ which does not appear in $W_{(\delta,f)}$. The argument is similar to the one used for the Hecke algebra of type A, see, e.g., [31, Lemma 2.11(b)], thus we omit it here. □

Theorem 2.7. *The algebra $\mathcal{A}_f = \mathrm{End}_{U_q\mathfrak{so}_N}(V_{m\epsilon} \otimes V^{\otimes f})$ is generated by the elements T and $\check{R}_i$, $i = 1, 2, \ldots, f-1$, where we assume $m = 1$ for N odd, and $m \in \mathbb{N}$ for N even. Moreover, we have*:

(a) *The algebra $\mathcal{A}_f$ decomposes as $\mathcal{A}_f \cong \overline{H}_f \oplus \langle \mathcal{A}_{f-1}, e_{A_{f-1}} \rangle$ where $\overline{H}_f$ is a quotient of the Hecke algebra of type B_f and the second summand is Jones' basic construction for $\mathcal{A}_{f-2} \subset \mathcal{A}_{f-1}$.*
(b) *The dimension of a simple $(\mathcal{A}_f)_\lambda$-module can be computed inductively by the restriction rule given by the tensor product rules for orthogonal groups.*
(c) *The weights of the Markov trace are given by*

$$\left(\dim_q V_\lambda / \dim_q V_{m\epsilon} (\dim_q V)^f\right)_{\lambda \in \Lambda_f(N,m)}.$$

Proof. The proof of this theorem follows the same pattern which was used already in [28]. It goes by induction on f with $f = 0$ being trivially true and $f = 1$ already shown in Proposition 2.6. Assume that $\widetilde{\mathcal{A}}_j = \mathcal{A}_j$ for $j = 0, 1, \ldots, f$. Observe that V is self-dual, i.e. $V \cong V^*$. Hence we can apply Proposition 1.4 with $X = V_{m\epsilon} \otimes V^{\otimes f-1}$, and with e being equal to $E_f / \dim_q V$ (see remarks at the beginning of Section 2.2). Then it follows from Proposition 1.5 that $\mathrm{End}_{U_q\mathfrak{g}}(V_{m\epsilon} \otimes V^{\otimes f+1})_{\mathrm{old}}$ is given by $\mathcal{A}_f E_f \mathcal{A}_f$. Moreover, by Proposition 2.6 also the restrictions of $\mathcal{A}_f$ and $\widetilde{\mathcal{A}}_f$ to $(V_{m\epsilon} \otimes V^{\otimes f+1})_{\mathrm{new}}$ coincide. As $\mathcal{A}_{f+1}$ modulo the ideal generated by E_f satisfies the relations of the Hecke algebra of type B_{f+1}, its action on $(V_{m\epsilon} \otimes V^{\otimes f+1})_{\mathrm{new}}$ factors through HB_{f+1}. As both quotient and ideal are semisimple, the Hecke algebra quotient splits as a direct summand. □

3. Generic weight formulas

In this section we will find general formulas for the q-dimension of irreducible representations of $\mathfrak{so}_{2n}$, for all n, whose number of factors will not depend on n. This was done explicitly for $\mathfrak{so}_{2n+1}$ in [28] for representations which exponentiate to representations of $SO(2n+1)$, i.e. whose highest weights are given by vectors with integer entries; the analogous statement for $\mathfrak{so}_{2n+1}$ follows easily from that (see Lemma 3.1 below). Here we derive analogous results for representations whose highest weights are given by vectors whose coordinates are half-integers. Let us first briefly review the results in [28], as they will be needed here.

Let λ be a Young diagram. As usual, we refer to the box in the ith row and jth column for λ by the ordered pair (i, j). The hook length $h(i, j)$ for a Young diagram λ is defined by $h(i,j) = \lambda_i - i + \lambda'_j - j + 1$. We also need the quantities $d(i,j)$, defined by

$$d_\lambda(i,j) = \begin{cases} -\lambda'_i - \lambda'_j + i + j - 1 & \text{if } i \geqslant j, \\ \lambda_i + \lambda_j - i - j + 1 & \text{if } i < j. \end{cases}$$

For fixed n, we shall also consider the following functions which were defined in [28]:

$$Q_\lambda(q) = \prod_{\substack{(i,j)\in\lambda \\ i\neq j}} \frac{[2n-1+d_\lambda(i,j)]_q}{[h(i,j)]_q} \times \prod_{(j,j)\in\lambda} \frac{[2n-1+\lambda_j-\lambda'_j]_q + [h(j,j)]_q}{[h(j,j,)]_q}.$$

It is obvious from this formula that we can define a rational function $Q_\lambda(r,q)$ depending on two variables r and q by substituting, for $m \in \mathbb{N}$, $[2n-1+a]$ by $(rq^a - r^{-1}q^{-a})/(q-q^{-1})$ in the formula above. Hence we obtain

$$Q_\lambda(r,q) = \prod_{\substack{(i,j)\in\lambda \\ i\neq j}} \frac{rq^{d_\lambda(i,j)} - r^{-1}q^{-d_\lambda(i,j)}}{q^{h(i,j)} - q^{-h(i,j)}} \times \prod_{(j,j,)\in\lambda} \frac{rq^{\lambda_j-\lambda'_j} - r^{-1}q^{-\lambda_j+\lambda'_j} + q^{h(j,j,)} - q^{-h(j,j)}}{q^{h(j,j)} - q^{-h(j,j)}}. \tag{13}$$

Lemma 3.1. *In the following we identify a dominant integral weight λ of type BCD with integer coefficients* (*as, e.g., in the notation of Section* 1.2) *with a Young diagram in the obvious way*:

(a) *If* $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $\dim_q V_\lambda = Q_\lambda(q^{2n}, q)$.
(b) *If* $\mathfrak{g} = \mathfrak{so}_{2n}$, $\dim_q V_\lambda = Q_\lambda(q^{2n-1}, q)$, *if* $\lambda_n = 0$.
(c) *If* $\mathfrak{g} = \mathfrak{sp}_{2n}$, $\dim_q V_\lambda = Q_\lambda(-q^{2n+1}, q)$.

Proof. The statement has been proved for type B in [28, Section 5] (which led to the derivation of the functions $Q_\lambda(r,q)$); the analogous results for types C and D will be proved at the end of Section 4.1. □

The formula for Q_λ (for $\mathfrak{so}_{2n}$) can also be rewritten as

$$Q_\lambda(q) = \prod_{\substack{(i,j)\in\lambda \\ i\neq j}} \frac{[2n-1+d_\lambda(i,j)]_q}{[h(i,j)]_q} \times \prod_{(j,j)\in\lambda} \frac{[n+\lambda_j-j]_q\bigl(q^{n-1+j-\lambda'_j} + q^{-n+1-j+\lambda'_j}\bigr)}{[h(i,j)]_q}. \tag{14}$$

Recall that the q-dimension of the highest weight $U_q\mathfrak{so}_{2n}$-module V_λ is given by

$$\dim_q V_\lambda = \prod_{1\leqslant i<j\leqslant n} \frac{[\lambda_i-\lambda_j+j-i]_q[2n+\lambda_i+\lambda_j-i-j]_q}{[j-i]_q[2n-i-j]_q}. \tag{15}$$

We will use the following equality in our computations below (see, e.g., [19]):

$$\prod_{1\leqslant i<j\leqslant n} \frac{[\lambda_i-\lambda_j+j-i]_q}{[j-i]_q} = \prod_{(i,j)\in\lambda} \frac{[n+j-i]_q}{[h(i,j)]_q}. \tag{16}$$

We shall also need the following observation, where α is a Young diagram:

$$\prod_{(j,j)\in\alpha} \frac{[n+\alpha_j-j]_q}{[n-1-\alpha'_j+j]_q} = \prod_{(i,j)\in\alpha} \frac{[n+j-i]_q}{[n+j-i-1]_q}. \tag{17}$$

To see that this is true observe that

$$\prod_{i=1}^{l(\alpha)} \frac{[n+\alpha_i-i]_q}{[n-i]_q} = \prod_{(i,j)\in\alpha} \frac{[n+j-i]_q}{[n+j-i-1]_q} = \prod_{j=1}^{\alpha_1} \frac{[n+j-1]_q}{[n-1-\alpha'_j+j]_q};$$

indeed, we obtain the equality of the first and second expression by taking the products of the boxes of each row separately; the second equality follows similarly by taking products over columns. Now we can cancel each factor in the first expression for which $\alpha_i < i$ with a factor in the denominator. Similarly, factors for which $\alpha'_j < j$ cancel with factors in the numerator of the third expression. After carrying out these cancellations, the first and the third expression become products where the numerators in both cases only consist of factors of the form $[n+y]$ with $y>0$ and the denominators consist of factors of the form $[n+y]$ with $y<0$. Hence they have to be the same (where n is viewed as a variable).

The last observation we need is as follows: It is well known that the sets $\{\lambda_i - i\colon i=1,\ldots,\lambda'_1\}$ and $\{i-1-\lambda'_i\colon i=1,\ldots,\lambda_1\}$ are mutually disjoint and the smallest number in the union of these two sets is $-\lambda'_1$ (these are the differences of x-coordinate minus y-coordinates of the corners of λ). Let $c=\max\{i\colon \lambda_i\geqslant i\}$. It is easy to see that the negative numbers in these 2 sets are given by $\{\lambda_i-i\colon i=c+1,\ldots,\lambda'_1\}$ and $\{i-1-\lambda'_i\colon i=1,\ldots,c\}$. Hence we obtain

$$\begin{aligned}&\bigl\{\lambda_i-i\colon i=c+1,\ldots,\lambda'_i\bigr\}\cap\bigl\{i-1-\lambda'_i\colon i=1,\ldots,c\bigr\}\\ &\quad=\bigl\{-1,-2,\ldots,-\lambda'_1\bigr\}.\end{aligned} \tag{18}$$

For the connection between λ and α, viewed as Young diagrams, see Fig. 9.

Lemma 3.2. *If* $\lambda=(m/2+\alpha_1,\ldots,m/2+\alpha_{l(\alpha)},m/2,\ldots,m/2)$, *then*

$$\frac{\dim_q V_\lambda}{\dim_q V_{m\epsilon}} = Q_{(\alpha,\emptyset)} = \prod_{(i,j)\in\alpha} \frac{[n+j-i]_q[2n-1+m+d(i,j)]_q}{[h(i,j)]_q[n+m+j-i-1]_q}$$

where

$$d(i,j)=\begin{cases}\alpha_i+\alpha_j-j-i+1 & \text{if } i>j,\\ -\alpha'_i-\alpha'_j+j+i-1 & \text{if } i\leqslant j.\end{cases}$$

Proof. The proof is by induction on m. For $m=0$, it follows from Eq. (15), Lemma 3.1(b) and Eq. (14) that

$$\begin{aligned}\dim_q V_\lambda &= \prod_{1\leqslant i<j\leqslant n}\frac{[\alpha_i-\alpha_j+j-i]_q[2n+\alpha_i+\alpha_j-i-j]_q}{[j-i]_q[2n-i-j]_q}\\
&=\prod_{\substack{(i,j)\in\alpha\\ i\neq j}}\frac{[2n-1+d(i,j)]_q}{[h(i,j)]_q}\\
&\quad\times\prod_{(j,j)\in\alpha}\frac{[2n-1+\alpha_j-\alpha'_j]_q+[h(j,j)]_q}{[h(j,j,)]_q}\\
&=\prod_{(i,j)\in\alpha}\frac{[2n-1+d(i,j)]_q}{[h(i,j)]_q}\\
&\quad\times\prod_{(j,j)\in\alpha}\frac{[n+\alpha_j-j]_q(q^{n-1+j-\alpha'_j}+q^{-n+1-j+\alpha'_j})}{[2n-2-2\alpha'_j+2j]_q}\\
&=\prod_{(i,j)\in\alpha}\frac{[2n-1+d(i,j)]_q}{[h(i,j)]_q}\prod_{(j,j)\in\alpha}\frac{[n+\alpha_j-j]_q}{[n-1-\alpha'+j]_q}.\end{aligned}$$

Therefore, by Eq. (17) we have our result:

$$\dim_q V_\lambda=\prod_{(i,j)\in\alpha}\frac{[n+j-i]_q[2n-1+d(i,j)]_q}{[h(i,j)]_q[n+j-i-1]_q}.$$

If $m=1$ we use Lemma 3.1(a). The q-dimension of an $\mathfrak{so}_{2n+1}$-module V_α, where α is an integer weight of $\mathfrak{so}_{2n+1}$, can be written explicitly as

$$\prod_{i=1}^{\alpha'_1}\frac{[n+1/2+\alpha_i-i]_q}{[n+1/2-i]_q}\\
\times\prod_{1\leqslant i<j\leqslant n}\frac{[2n+1+\alpha_i+\alpha_j-i-j]_q[\alpha_i-\alpha_j+j-i]_q}{[2n+1-i-j]_q[j-i]_q}.$$

By the already mentioned lemma, this is equal to $Q_\alpha(q^{2n},q)$, which we can write as

$$\prod_{(j,j)\in\alpha}\frac{[n+1/2+\alpha_j-j]_q}{[n+1/2+j-1-\alpha'_j]_q}\prod_{\substack{(i,j)\in\alpha\\ i\neq j}}\frac{[2n+d(i,j)]_q}{[h(i,j)]_q}.$$

Combining Eq. (17) and the equation below it, we obtain

$$\prod_{i=1}^{\alpha_1'} \frac{[n+1/2-i]_q}{[n+1/2+\alpha_i-i]_q} \prod_{(j,j)\in\alpha} \frac{[n+1/2+\alpha_j-j]_q}{[n+1/2+j-1-\alpha_j']_q} = 1.$$

Therefore, we have

$$\begin{aligned} \frac{\dim_q V_{\epsilon+\alpha}}{\dim_q V_\epsilon} &= \prod_{1\leqslant i<j\leqslant n} \frac{[2n+1+\alpha_i+\alpha_j-i-j]_q[\alpha_i-\alpha_j+j-i]_q}{[2n+1-i-j]_q[j-i]_q} \\ &= \prod_{(i,j)\in\alpha} \frac{[2n+d(i,j)]_q}{[h(i,j)]_q}. \end{aligned}$$

This proves the lemma for $m=1$. Let now $m \geqslant 2$. By induction assumption, the claim is true for $m-2$ and for all n. By Eq. (15) we have

$$Q_{(\alpha,\emptyset)} = \prod_{1\leqslant i<j\leqslant n} \frac{[\alpha_i-\alpha_j+j-i]_q[2n+m+\alpha_i+\alpha_j-i-j]_q}{[j-i]_q[2n+m-i-j]_q}.$$

By (16) it will suffice to show that

$$\begin{aligned} &\prod_{1\leqslant i<j\leqslant n} \frac{[2n+m+\alpha_i+\alpha_j-i-j]_q}{[2n+m-i-j]_q} \\ &\quad= \prod_{(i,j)\in\alpha} \frac{[2n-1+m+d(i,j)]_q}{[n+m+j-i-1]_q}. \end{aligned} \tag{$*$}$$

Note that we can rewrite the left-hand side of $(*)$ as

$$\begin{aligned} &\prod_{1\leqslant i<j\leqslant n+1} \frac{[2(n+1)+m-2+\alpha_i+\alpha_j-i-j]_q}{[2(n+1)+m-2-i-j]_q} \\ &\quad\times \prod_{i=1}^{n} \frac{[n+m-1-i]_q}{[n+m+\alpha_i-1-i]_q}. \end{aligned}$$

Since we know that our claim is true for $m-2$ and $n+1$, we use the inductive step for the first product in our expression (and some obvious cancellations for the second product) to obtain

$$\prod_{(i,j)\in\alpha} \frac{[2n-1+m+d(i,j)]_q}{[n+m-2-i+j]_q} \prod_{i=1}^{l(\alpha)} \frac{[n+m-1-i]_q}{[n+m+\alpha_i-1-i]_q}.$$

Observe that the terms for $j=1$ in the denominator of the first product cancel with the numerators of the second product. The denominator of the remaining product can be rewritten as

$$\prod_{i=1}^{l(\alpha)}[n+m+\alpha_i-1-i]_q\prod_{j=2}^{\alpha_i}[n+m-2-i+j]_q$$
$$=\prod_{(i,j)\in\alpha}[n+m+j-1-i]_q.$$

This proves our claim. □

Lemma 3.3. *If* $\lambda=(m/2,\dots,m/2,m/2-\beta_{l(\beta)},\dots,m/2-\beta_1)$, *then*

$$\frac{\dim_q V_\lambda}{\dim_q V_{m\epsilon}}=Q_{(\emptyset,\beta)}=\prod_{(i,j)\in\beta}\frac{[n+j-i]_q[m-1-d(i,j)]_q}{[h(i,j)]_q[n+m+i-j-1]_q},$$

where $d(i,j)$ *is as in the previous lemma.*

Proof. Observe that substituting $\lambda_i=m/2-\beta_{n+1-i}$ into Eq. (15) we obtain

$$\frac{\dim_q V_\lambda}{\dim_q V_{m\epsilon}}=\prod_{1\leqslant i<j\leqslant n}\frac{[\beta_{n+1-j}-\beta_{n+1-i}+j-i]_q}{[j-i]_q}$$
$$\times\frac{[2n+m-\beta_{n+1-i}-\beta_{n+1-j}-i-j]_q}{[2n+m-i-j]_q}.$$

Now reindex as follows: $i\to n+1-j$ and $j\to n+1-i$, and substitute m by $-2n-m'+2$. This gives

$$Q_{\lambda/m\epsilon}=\prod_{1\leqslant i<j\leqslant n}\frac{[\beta_i-\beta_j+j-i]_q[2n+m'-\beta_j-\beta_i+i+j]_q}{[j-i]_q[2n+m'+i+j]_q}.$$

By the previous lemma we have that

$$\prod_{1\leqslant i<j\leqslant n}\frac{[2n+m'+\beta_i+\beta_j-i-j]_q}{[2n+m'-i-j]_q}=\prod_{(i,j)\in\beta}\frac{[2n-1+m'+d(i,j)]_q}{[n+m'+j-i-1]_q}.$$

Substituting back m' by $-2n+2-m$ in this equation and multiplying both numerator and denominator by -1, we get

$$\prod_{1\leqslant i<j\leqslant n}\frac{[m-\beta_i-\beta_j+i+j-2]_q}{[m+i+j-2]_q}=\prod_{(i,j)\in\beta}\frac{[m-1-d(i,j)]_q}{[n+m+i-j-1]_q}.$$

Thus we have the desired result. □

We can now formulate the main result of this section. Its importance comes from the fact that the number of factors for the functions $\dim_q V_\lambda$ no longer depend on n.

Theorem 3.4. *Let $Q_{(\alpha,\emptyset)}$ and $Q_{(\emptyset,\beta)}$ be as in Lemmas* 3.2 *and* 3.3, *let* $\lambda = [m/2+\alpha_1,\ldots,m/2+\alpha_{l(\alpha)},m/2,\ldots,m/2,m/2-\beta_{l(\beta)},\ldots,m/2-\beta_1]$. *Then* $\dim_q V_\lambda/\dim_q V_{m\epsilon}$ *can be expressed only in terms of the Young diagrams α and β by the function $\widetilde{Q}_{(\alpha,\beta)}$ defined by*

$$\widetilde{Q}_{(\alpha,\beta)}(q) := Q_{(\alpha,\emptyset)}Q_{(\emptyset,\beta)}\prod_{i=1}^{l(\alpha)}\prod_{j=1}^{l(\beta)}\frac{[n+\alpha_i+\beta_j-i-j+1]_q}{[n+\alpha_i-i-j+1]_q} \times\frac{[n+m+\alpha_i-\beta_j+j-i-1]_q}{[n+m+\alpha_i+j-i-1]_q}$$
$$\times\prod_{i=1}^{l(\alpha)}\prod_{j=1}^{l(\beta)}\frac{[n-i-j+1]_q[n+m+j-i-1]_q}{[n+\beta_j-i-j+1]_q[n+m-\beta_j+j-i-1]_q}. \tag{19}$$

Proof. The proof consists of checking that $\dim_q V_\lambda/Q_{(\alpha,\emptyset)}Q_{(\emptyset,\beta)}$ is given by the two double products on the right-hand side. This is straightforward, and is left to the reader. $\square$

Similarly as in the beginning of this section, we derive a rational function in several variables from the formula in the previous theorem. We do this by replacing expressions of the form $[n+a]$ by $(r^{1/2}q^{a+1/2}-r^{-1/2}q^{-a-1/2})/(q-q^{-1})$ and $[n+m-1+a]$ by $(r'q^a-(r')^{-1}q^{-a})/(q-q^{-1})$. To prove theorems for the resulting functions, we use the following simple Zariski-density argument.

Lemma 3.5. *Let Q_1, Q_2 be two rational functions in three variables over the field F which contains infinitely many elements. Then $Q_1=Q_2$ if and only if the following holds*:

There exists an infinite subset $I\subset\mathbb{Z}$, and for each $n\in I$ there exists an infinite set J_n such that $Q_1(q,q^n,q^m)=Q_2(q,q^n,q^m)$ for each $n\in I$ and $m\in J_n$.

The following theorem now is an easy consequence of Theorem 3.4.

Theorem 3.6. *There exist for any pair (α,β) of Young diagrams a rational function $Q_{(\alpha,\beta)}$ in three variables q, r, r', obtained from $\widetilde{Q}_{(\alpha,\beta)}$ by the substitutions just described. $Q_{(\alpha,\beta)}$ no longer depends on n and m. Moreover, its zeros and poles all are either at q a root of unity or for r or r' being $\pm$ an integer power of q.*

Corollary 3.7. *Let (α,β) be a pair of Young diagrams. Then $\sum_{(\gamma,\delta)}Q_{(\gamma,\delta)} = xQ_{(\alpha,\beta)}$, where the summation goes over all pairs (γ,δ) obtained by adding or removing a box to/from one of the two diagrams α or β.*

Proof. If $r = q^{2n-1}$ and $r' = q^{2n+2m-2}$, with $n, m \gg f$, the claim follows from the branching rule, Eq. (6), and the formulas for the dimension functions (again, Fig. 9 may be helpful for translating weights into pairs of Young diagrams). $\square$

4. The BMW algebras

4.1. The BMW algebra of type A

The BMW algebra $\mathrm{BMW}_f(q, r)$ is the algebra over the field $\mathbb{C}(q, r)$ of rational functions in two variables r and q given by generators $g_1, \ldots, g_{f-1}$ which are assumed to be invertible and satisfy the following relations:

(1) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$;

(2) $g_i g_j = g_j g_i$ for $|i - j| > 1$;

(3) $e_i g_i = r^{-1} e_i$;

(4) $e_i g_{i-1}^{\pm 1} e_i = r^{\pm 1} e_i$;

where e_i is defined by the equation

$$\left(q - q^{-1}\right)(1 - e_i) = g_i - g_i^{-1}.$$

From this equation one obtains that e_i satisfies the relation

$$e_i^2 = x e_i \quad \text{where } x = \frac{r - r^{-1}}{q - q^{-1}} + 1.$$

One can also show using the defining relations that the g_i's satisfy the cubic relation

$$\left(g_i - r^{-1}\right)\left(g_i + q^{-1}\right)(g_i - q) = 0.$$

Thus the eigenvalues of g_i are r^{-1}, $-q^{-1}$ and q.

It was shown in [2] (see also [20]) that the ideal $I_f \subset \mathrm{BMW}_f$ generated by e_{f-1} is isomorphic to Jones' basic construction for $\mathrm{BMW}_{f-2} \subset \mathrm{BMW}_{f-1}$. Moreover, the quotient BMW_f / I_f is isomorphic to the Hecke algebra $H_f(q^2)$ and it splits. One deduces from this easily, using Theorem 1.1, that the irreducible representations of BMW_f are indexed by partitions of $f - 2k$ where $0 \leqslant k \leqslant \lfloor f/2 \rfloor$. The Bratteli diagram for the sequence of algebras $(\mathrm{BMW}_f)_f$ can be constructed by taking these partitions as the vertices. A partition λ in level f is connected by an edge to a partition μ in level $f + 1$, if μ can be obtained from λ by adding a box to λ or removing a box from λ.

There also exists a trace functional tr on BMW_f, the Markov trace, defined inductively by $\mathrm{tr}(1) = 1$, $\mathrm{tr}(g_i^{\pm 1}) = r^{\pm 1}/x$, where $x = 1 + (r - r^{-1})/(q - q^{-1})$, $i = 1, 2, \ldots, f - 1$, and by $\mathrm{tr}(a g_{f-1}^{\pm 1} b) = \mathrm{tr}(g_{f-1}^{\pm 1})\,\mathrm{tr}(ab)$ for $a, b \in \mathrm{BMW}_{f-1}$.

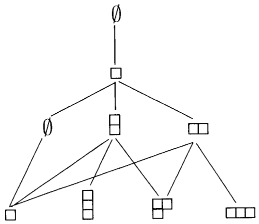


Fig. 8. Bratteli diagram for BMW algebra.

The existence of this trace was originally derived from knot theory, using the existence of the Kauffman link polynomial. We shall sketch a purely algebraic proof of this fact below; this will also serve as an outline for the proof of analogous statements for the B-BMW algebras, to be carried out later in this paper. The following theorem was proved in [28, Theorem 5.5].

Theorem 4.1. *The Markov trace* tr *on* BMW_f *has the weight vector* $(Q_\lambda(r,q)/x^f)_\lambda$; *here* $Q_\lambda(r,q)$ *is the rational function defined in Eq.* (13), λ *runs through all partitions of* $f-2k$, *where* $0 \leqslant k \leqslant \lfloor f/2 \rfloor$, *and* $x = 1 + (r - r^{-1})/(q - q^{-1})$.

The importance of the algebras BMW_f comes from the fact that one obtains homomorphisms of certain specializations of them into $\mathrm{End}(V^{\otimes n})$ via R-matrices; here V is the vector representation of an orthogonal or symplectic quantum group. More precisely, if $\mathfrak{g} = \mathfrak{so}_k$, the braid representation given by the R-matrices (see Eq. (3)) factors through $\mathrm{BMW}_f(q^{k-1}, q)$, while for $\mathfrak{g} = \mathfrak{sp}_k$, it factors through $\mathrm{BMW}_f(-q^{k+1}, q)$.

Proof of Lemma 3.1. It was already shown in Section 2.2 that $g_i \mapsto \check{R}_i$ defines a homomorphism from $\mathrm{BMW}_f(q^{2n-1}, q)$ into $\mathrm{End}_{U_q\mathfrak{so}_{2n}}(V^{\otimes f})$. Similarly, one obtains homomorphisms from $\mathrm{BMW}_f(q^{2n}, q)$ into $\mathrm{End}_{U_q\mathfrak{so}_{2n+1}}(V^{\otimes f})$ and from $\mathrm{BMW}_f(-q^{2n+1}, q)$ into $\mathrm{End}_{U_q\mathfrak{sp}_{2n}}(V^{\otimes f})$ (this has been observed before by a number of authors, including Reshetikhin and Turaev [25]). Let tr_q be the normalization of the functional Tr_q, defined on $\mathrm{End}(V^{\otimes f+1})$. It follows from Proposition 1.4, (b) and (c), that $\mathrm{tr}((a \otimes 1)(1_{f-1} \otimes \check{R}_f)) = \mathrm{tr}(a)\,\mathrm{tr}(\check{R}_f)$. One deduces that tr_q pulls back to the Markov trace tr under the homomorphisms described above. Comparing tensor product rules with the restriction rule for BMW_f, one checks that $\mathrm{End}_{U_q\mathfrak{g}}(V^{\otimes f})$ has the same dimension as BMW_f for $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and for $\mathfrak{g} = \mathfrak{sp}_{2n}$ for all $f \in \mathbb{N}$; the same is true for $\mathfrak{g} = \mathfrak{so}_{2n}$ as long as $f < n$. $\square$

4.2. The BMW algebra of type B

This algebra was first defined by Häring-Oldenburg [9]. The BMW algebra of type B, $BB_f(q, r, R)$ is the algebra over the field $\mathbb{C}(q, r, R)$ of rational functions in three variables r, q, and r', defined by generators $t, g_1, \dots, g_{f-1}$ which satisfy relations (1)–(4) as the BMW algebra of type A and in addition the following relations:

(5) $tg_1tg_1 = g_1tg_1t$,

(6) $t^2 = (r'+1)t - r'$,

(7) $g_it = tg_i$ for $i > 1$,

(8) $tg_1te_1 = r'q^{-1}e_1$,

(9) $e_1te_1 = \dfrac{(r'-1)(r-q^{-1})}{q-q^{-1}}e_1$.

Lemma 4.2. *Define $t_1 = t$ and $t_{i+1} = g_i \cdots g_1tg_1 \cdots g_i$ for $i > 1$. Then we have*

(a) $g_ft_fe_f = q^{-1}r't_f^{-1}e_f$.

(b) *Let $y \in BB_{f-1}$ be such that there exists an element $\varepsilon_{f-2}(y) \in BB_{f-2}$ satisfying $e_{f-1}ye_{f-1} = \varepsilon_{f-2}(y)e_{f-1}$. Then*

$$e_fg_{f-1}yg_{f-1}^{-1}e_f = e_fg_{f-1}^{-1}yg_{f-1}e_f = e_f\varepsilon_{f-2}(y).$$

(c) $e_ft_fe_f = \varepsilon_f(t_f)e_f$, *where one can define $\varepsilon_i(t_i) \in BB_i$ inductively by $\varepsilon_1(t) =$ $\mathrm{tr}(t)$ and $\varepsilon_i(t_i) = \varepsilon_{i-1}(t_{i-1}) + (q-q^{-1})(rt_{i-2} - q^{-1}r't_{i-2}^{-1})$.*

(d) $t_{f+1}e_f = r'r^{-1}q^{-1}t_f^{-1}e_f$.

Proof. The proofs go by induction on f. Observe that $t_f = g_ft_{f-1}g_f$. For (a), the claim follows for $f = 1$ from relation (8). Using $g_{f-1}^{\pm 1}e_f = g_f^{\mp 1}e_{f-1}e_f$ and $g_fg_{f-1}g_f^{-1} = g_{f-1}^{-1}g_fg_{f-1}$, we obtain

$$\begin{aligned} g_ft_fe_f &= g_fg_{f-1}t_{f-1}g_f^{-1}e_{f-1}e_f = g_{f-1}^{-1}g_fg_{f-1}t_{f-1}e_{f-1}e_f \\ &= q^{-1}r'g_{f-1}^{-1}g_ft_{f-1}^{-1}e_{f-1}e_f = q^{-1}r't_f^{-1}e_f. \end{aligned}$$

The l.h.s. of statement (b) is equal to

$$e_fg_{f-1}g_fyg_f^{-1}g_{f-1}e_f = e_fe_{f-1}ye_{f-1}e_f = \varepsilon_{f-2}(y)e_f,$$

using $e_fe_{f-1}e_f = e_f$. Using $g_{f-1} = g_{f-1}^{-1} + (q-q^{-1})(1-e_{f-1})$, we obtain for (c)

$$\begin{aligned} e_ft_fe_f &= e_f\big[g_{f-1}t_{f-1}\big(g_{f-1}^{-1} + \big(q-q^{-1}\big)(1-e_{f-1})\big)\big]e_f \\ &= e_f\varepsilon_{f-1}(t_{f-1}) + \big(q-q^{-1}\big)\big(re_ft_{f-1} - q^{-1}r'e_ft_{f-1}^{-1}e_{f-1}e_f\big). \end{aligned}$$

From this follows the claim for (c). Part (d) can be easily reduced to part (a) using the inductive formula for t_f. □

Proposition 4.3. (a) $BB_{f+1}e_f = BB_f e_f$.

(b) *Any element of BB_{f+1} which is in the ideal generated by e_f can be written as a linear combination of elements of the form $ae_f b$, with $a, b \in BB_f$.*

(c) *Any element in BB_{f+1} can be written as a linear combination of elements of the form $a\chi b$, with $a, b \in BB_f$ and $\chi \in \{1, g_f, e_f, t_{f+1}\}$. In particular, BB_{f+1} is finite dimensional for all f.*

Proof. The claims are shown by induction on f. Part (a) follows from Lemma 4.2, using the induction assumption from part (c). Part (b) is shown similarly by observing that the proofs of Lemma 4.2 also work for simplifying expressions obtained by multiplying generators from the right. Hence statement (c) also holds for elements in the ideal generated by e_f. The quotient of BB_{f+1} modulo this ideal is isomorphic to the Hecke algebra of type B_{f+1}. For these algebras the claim has already been shown in [8] (where our t_i is denoted by t_i'). □

In order to show that the inclusion $BB_f \mapsto BB_{f+1}$ is injective, we will use the algebras $\mathcal{A}_f$.

Lemma 4.4. *The map $g_i \mapsto \check{R}_i$, $t \mapsto T$ induces a homomorphism of the specialization $BB_f(q, q^{2n-1}, q^{2n+2m-2})$ into $\mathcal{A}_f$.*

Proof. It suffices to check that relations (1)–(9) are preserved. This follows from Proposition 2.4 for all relations except (8). This relation is probably easiest checked using the tangle formalism (see [14,27]), or, equivalently, the braiding axioms. We give here an algebraic proof. Observe that both $V_{m\epsilon+\varepsilon_1} \otimes V$ and $V_{m\epsilon-\varepsilon_n} \otimes V$ contain a submodule isomorphic to $V_{m\epsilon}$ with multiplicity one. Hence $V_{m\epsilon}$ appears in $V_{m\epsilon} \otimes V^{\otimes 2}$ with multiplicity 2 and the highest weight vectors of these two submodules span a 2-dimensional $\mathcal{A}_2$-module. Moreover, in this module, $\check{R}_1$ has eigenvalues q^{1-2n} and $-q^{-1}$, and its action coincides with the one of

$$-q^{-1}1 + \frac{r^{-1}(1+q^{-1})}{r - r^{-1} + q - q^{-1}} E_1.$$

Substituting this expression into relation (8) for g_1 and using relations (7) and (8) of Proposition 2.4 we arrive at the claim. □

In order to relate our results about $\mathcal{A}_f$ to BB_f we shall need a different labeling set for the simple components of $\mathcal{A}_f$. Let us introduce the following notation (where α, β are Young diagrams):

$$\Gamma(2n, m) = \bigl\{(\alpha, \beta)\colon \alpha_1' + \beta_1' \leqslant n,\ \beta_1 + \beta_2 \leqslant m,\ \beta_2 < m/2\bigr\},$$

and

$$\Gamma_f(2n,m) = \big\{(\alpha,\beta) \in \Gamma(2n,m)\colon\ |\alpha| + |\beta| = f - 2i,\ i = 0, 1, \ldots\big\}.$$

Lemma 4.5. *There exists a* 1–1 *correspondence between the elements* $\Gamma_f(2n,m)$ *as defined here, and the ones of* $\Lambda_f(2n,m)$ *as defined in Section* 2.1*, for* $n, m \gg f$*. In particular, we have*

(a) *The simple components of* $(V_{m\epsilon} \otimes V^{\otimes f})_{\mathrm{new}}$ *are labeled by the elements* $(\alpha,\beta) \in \Gamma(2n,m)$ *for which* $|\alpha| + |\beta| = f$*.*
(b) *Let* $W_{(\alpha,\beta,f)}$ *denote a simple* $\mathcal{A}_f$*-module. Then its dimension can be computed inductively by the restriction rule*

$$W_{(\alpha,\beta,f)} \cong \bigoplus_{(\alpha',\beta',f-1)} W_{(\alpha',\beta',f-1)},$$

where the summation goes over all pairs (α',β') *which can be obtained by removing/adding one box from/to one of the diagrams* α *or* β*.*
(c) *For* $n, m \gg f$*, the set* $\Gamma_f(2n,m)$ *does not depend on the particular choice of* n *and* m*, and will be denoted by* Γ_f*. It consists of all pairs* (α,β) *for which* $f - |\alpha| - |\beta|$ *is nonnegative and even.*

Proof. Recall the description of the set P^+ of dominant integral weights of $\mathfrak{so}_{2n}$ from Section 1.2. The bijection between elements of P^+ and of $\Lambda(2n,m)$ follows from Fig. 9, where β is obtained by rotating β^* by 180 degrees. The restriction rule follows from the branching rule by induction, where adding $\pm\varepsilon_i$ corresponds to adding or removing a box from the given diagram. This proves (b). In particular, we also obtain the correct description of the labeling set from the branching rule by induction, from which we get (a). $\square$

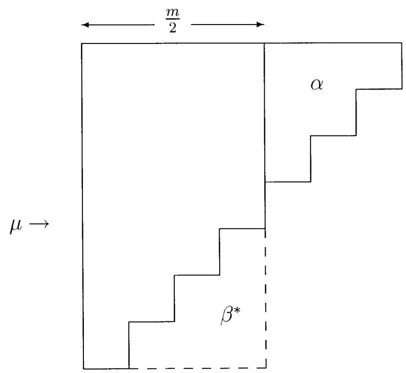


Fig. 9.

In the following, we want to determine the structure of $BB_f(q,r,r')$. We thereby reprove results obtained by Häring-Oldenburg; in addition, we can also determine for which values of q, r, r' the algebra $BB_f(q,r,r')$ is semisimple.

For the next lemma, we will assume that $BB_{f-1} \subset BB_f$ has the same inclusion diagram and the same dimensions as $\mathcal{A}_{f-1} \subset \mathcal{A}_f$, with n and m large. This is certainly satisfied for $f = 1$, with BB_0 and $\mathcal{A}_0$ being the 1-dimensional algebras over their respective ground fields, and BB_1 and $\mathcal{A}_1$ being the direct sum of two copies over their respective ground fields. Observe that the functions in Section 3 can now be used to define a trace functional tr over BB_f, whose restriction to BB_{f-1} is given by the corresponding weight functions. As these weight functions are nonzero, also the restriction of tr to BB_{f-1} is nondegenerate. Let ε_{f-1} be the conditional expectation defined by tr.

Lemma 4.6. *With the assumptions just stated, and with $x = 1 + (r - r^{-1})/(q - q^{-1})$, we have*

(a) $e_f b e_f = x\varepsilon_{f-1}(b)e_f$ *for all* $b \in BB_f$.
(b) *The inclusion $BB_f \to BB_{f+1}$ is injective.*
(c) *The ideal generated by e_f in BB_{f+1} is isomorphic to Jones' basic construction for $BB_{f-1} \subset BB_f$ with respect to* tr.

Proof. We prove the lemma by induction on f, with the claims being trivially true for $f = 0$ and $f = 1$. Observe that for specializations $r = q^k$, $r' = q^l$ with k and l sufficiently large, for which we have a homomorphism into $\mathcal{A}_f$ for suitable choices of n and m, we have $e_f b e_f(q, q^k, q^l) = (x\varepsilon_{f-1}(b))(q, q^k, q^l)$. Indeed, this is known for its image in $\mathcal{A}_f$, and hence follows for its pull-back, which does not change traces or conditional expectations. Let now b' be an expression in the generators of BB_{f-1} such that $e_f b e_f = xb'e_f$, which exists by Proposition 4.3. Then $\mathrm{tr}(db')(q, q^k, q^l) = \mathrm{tr}(d\varepsilon_{f-1}(b))(q, q^k, q^l)$ for all $d \in BB_{f-1}$, and all k, l sufficiently large. Hence $b' = \varepsilon_{f-1}(b)$ by Lemma 3.5, which shows (a).

It follows from Lemma 4.2 that there exists, for any $b \in BB_f$ an element $b_g \in BB_f$ such that $g_f b e_f = b_g e_f$. Let us show that b_g does not depend on which relations we used to get this equality. This is done as before by checking the claim for a Zariski-dense subset of the parameters. Let $\tilde{b}_g \in BB_f$ be another element such that $\tilde{b}_g e_f = g_f b e_f$. We know that $b_g = \tilde{b}_g$ for $r = q^k$, $r' = q^l$, for k, l sufficiently large. Hence

$$\mathrm{tr}(db_g)\big(q, q^k, q^l\big) = \mathrm{tr}\big(d\tilde{b}_g\big)\big(q, q^k, q^l\big) \quad \text{for all } d \in BB_{f-1}.$$

Hence $\mathrm{tr}(db_g)$ and $\mathrm{tr}(d\tilde{b}_g)$ describe the same rational functions for any $d \in BB_f$, by Lemma 3.5, and we get $b_g = \tilde{b}_g$ from the nondegeneracy of tr. This shows that the action of g_f on BB_f is well-defined.

It remains to check that the linear operator defined by g_f is compatible with the relations for BB_{f+1}. E.g. for checking the braid relations, it suffices to show that

$$\mathrm{tr}\big(d\big[(g_f g_{f-1} g_f)b\big]\big) = \mathrm{tr}\big(d\big[(g_{f-1} g_f g_{f-1})b\big]\big)$$

for all $b, d \in BB_f$; here $[(g_f g_{f-1} g_f)b]$ and $[(g_{f-1} g_f g_{f-1})b]$ are elements in BB_f defined by the action of g_f and left multiplication by g_{f-1}. This can be shown in exactly the same way as the well-definedness of the action of g_f, using the fact that we already know equality for the specializations $\mathcal{A}_f$. The other relations are checked in the same way. Hence we have obtained a representation of BB_{f+1} whose restriction to BB_f is injective. This implies (b).

It follows from (b) that the conditions for Theorem 1.1 are satisfied for the inclusion $BB_{f-1} \subset BB_f$ together with the projection $e = (1/x)e_f$. So the ideal generated by e_f in the algebra generated by BB_f and e_f is isomorphic to Jones' basic construction for $BB_{f-1} \subset BB_f$. On the other hand, this ideal coincides with the ideal generated by e_f in BB_{f+1}, by Proposition 4.3(b). Statement (c) is proved. □

We can now prove the following theorem, all of whose statements, except for the second sentence in (c) can be found in work of Häring-Oldenburg, see [9].

Theorem 4.7. (a) *The algebra BB_f decomposes as $BB_f \cong H_f \oplus \langle BB_{f-1}, e_{BB_{f-1}}\rangle$ where H_f is the generic Hecke algebra of type B and the second summand is Jones' basic construction for $BB_{f-2} \subset BB_{f-1}$.*

(b) *The algebra BB_f is semisimple, with the simple components labeled by the elements in Γ_f. The dimension of a simple $(BB_f)_{(\alpha,\beta)}$-module can be computed inductively by the restriction rule.*

(c) *There exists a well-defined faithful trace* tr *on the inductive limit of the algebras BB_f such that $\mathrm{tr}(g_f^{\pm 1} b) = \mathrm{tr}(g_f^{\pm 1})\,\mathrm{tr}(b)$ for all $b \in BB_f$. If p is a minimal idempotent in $(BB_f)_{(\alpha,\beta)}$, $\mathrm{tr}(p) = Q_{(\alpha,\beta)}/x^f$, with $x = 1 + (r - r^{-1})/(q - q^{-1})$, and with $Q_{(\alpha,\beta)}$ as in Theorem* 3.6.

Proof. The theorem is proved by induction on f with $f = 0$ and $f = 1$ being trivially true. Recall that the quotient of BB_{f+1} modulo the ideal generated by e_f is isomorphic to the Hecke algebra HB_{f+1} of type B. On the other hand, we obtain a representation of BB_{f+1} on BB_f in the previous lemma. Both representations are semisimple. Moreover, on each simple BB_{f+1}-submodule of BB_f the element e_f acts nonzero. Hence it cannot be a simple HB_f-module, and the Hecke algebra quotient has to split. Moreover, we already know that the dimensions of the basic construction part and the Hecke algebra add up to $2^n n!!$, from which we get (a). Also, part (b) follows from the restriction rules for the basic construction and the Hecke algebras of type B.

To prove part (c), we extend the trace tr via our weight formulas to BB_{f+1}. This is well-defined by Corollary 3.7 (see also Section 1.1). Observe that this trace is well-defined for all specializations $r = q^{2n-1}$, $r' = q^{2m+2n-2}$ for which the functions $Q_{(\alpha,\beta)}$ are nonzero for all $(\alpha, \beta) \in \Lambda_f \cup \Lambda_{f-1}$. It is easy to see that for n and m sufficiently large, these conditions are satisfied. As tr satisfies the Markov property for all these specializations, it must hold as well for the generic case. □

We obtain, as a corollary of the proof of the theorem above and of Theorem 3.6, the following theorem.

Theorem 4.8. *Let $\zeta, \rho, \rho' \in \mathbb{C}$, and let $BB_f(\zeta, \rho, \rho')$ be the complex algebra defined as before BB_f with substitutions $q = \zeta$, $r = \rho$, and $r' = \rho'$. Then $BB_f(\zeta, \rho, \rho')$ is semisimple for all values ζ, ρ, ρ' for which ζ is not a root of unity, and for which $Q_{(\alpha,\beta)}(\zeta, \rho, \rho') \neq 0$ for all $(\alpha, \beta) \in \Lambda_f \cup \Lambda_{f-1}$. This is the case, in particular, if ρ and ρ' are not equal to $\pm$ a power of ζ.*

5. Classical limit: $q \to 1$

In this section we assume q to be a complex number. We consider the limit of the BMW algebra of type B for r and r' powers of q as $q \to 1$. We obtain a 'degenerate' version of our algebra, similarly as one obtains a degenerate affine Hecke algebra from the affine Hecke algebra of type A (see [4,7]), by essentially the same method.

5.1. Degenerate Hecke algebra of type B

Definition 5.1. The degenerate Hecke algebra HB_f^d of type B_f depends on a parameter k and is given by generators $1, p, \tilde{s}_1, \tilde{s}_2, \ldots, , \tilde{s}_{f-1}$ and relations

(1) $\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_{f-1}$ satisfy the relations of simple reflections of the symmetric group S_f;
(2) $p^2 = kp$;
(3) $p\tilde{s}_1 p\tilde{s}_1 + 2p\tilde{s}_1 = \tilde{s}_1 p\tilde{s}_1 p + 2\tilde{s}_1 p$ and $p\tilde{s}_i = \tilde{s}_i p$ for $i > 1$.

As reviewed in Section 2.2, Hoefsmit [10] defined explicit matrix representations for the Hecke algebra of type B indexed by pairs of Young diagrams. He wrote matrices with entries in $\mathbb{C}(q, r')$ for each generator $t, g_1, \ldots, g_{f-1}$. The images of g_i, $i = 1, 2, \ldots, f-1$, are well-defined at $q = 1$ and satisfy the relations of simple reflections of the symmetric group S_f. The image of t is given by a diagonal matrix with eigenvalues r' and 1. Setting $r' = q^k$, for k a formal parameter, and differentiating with respect to q at $q = 1$, we obtain a matrix with eigenvalues

k and 0. It is now easy to check that this matrix together with the images of g_i at $q = 1$ satisfy the relations of the degenerate Hecke algebra HB_f^d (observe that relation (3) can be checked within HB_2^d, which has four 1-dimensional and one 2-dimensional representation). Thus, for every representation of $HB_f(q, q^k)$ we get a representation of HB_f^d.

Proposition 5.2. *The degenerate Hecke algebra HB_f^d has the same dimension and the same decomposition into simple matrix rings as the Hecke algebra of type B_f.*

Proof. It follows from the discussion before this theorem that we obtain a representation of HB_f^d for each representation of the Hecke algebra $HB_f(q, q^k)$. It is not hard to deduce from this the existence of a collection of mutually non-isomorphic irreducible representations of HB_f^d, labeled by ordered pairs of Young diagrams (α, β) for which $|\alpha| + |\beta| = f$; the proof follows the one for $HB_f(q, q^k)$ word for word (see, e.g., Proposition 2.6). Hence $\dim HB_f^d \geqslant \dim HB_f = 2^f f!$.

The proposition is proved as soon as it is shown that $\dim HB_f^d \leqslant \dim HB_f$. To do so, we define elements $p_j = \tilde{s}_j \cdots \tilde{s}_1 p \tilde{s}_1 \cdots \tilde{s}_j$ for $j = 0, 1, 2, \ldots, f - 1$, where $p_0 = p$. Moreover, we define the set $\mathcal{R}_i = \{\tilde{s}_{f-1} \cdots \tilde{s}_i, \tilde{s}_{f-1} \cdots \tilde{s}_i p_{i-1}$: $1 \leqslant i \leqslant f\}$ (where $\mathcal{R}_1 = \{1, p\}$). We claim that a spanning set S of HB_f^d is given by $\{r_1 \cdots r_f\colon\ r_i \in \mathcal{R}_i\}$. As S contains the generators of HB_f^d, it suffices to show that multiplication of an element in S by a generator results in a linear combination of elements in S. This is straightforward to check. Alternatively, it can also be deduced from a similar statement for the Hecke algebra $HB_f(q, q^k)$: it is known that one obtains a basis of the Hecke algebra $HB_f(q, q^k)$ by replacing the elements in our set above by the elements in $HB_f(q, q^k)$ obtained by substituting each occurrence of $\tilde{s}_i$ by g_i, and each p_j by $t'_j = g_j \cdots g_1 t g_1^{-1} \cdots g_j^{-1}$ (see [8]). As the relations of HB_f^d are the same as the ones of HB_f after these substitutions, modulo words of shorter length (compare with Section 1.5), the proof for HB_f^d can be deduced from the one for $HB_f(q, q^k)$. □

Now we compare this algebra with the *degenerate affine Hecke algebra*. This algebra is generated by the group algebra $\mathbb{C}[S_n]$ and the pairwise commuting elements $v_1, v_2, \ldots, v_n$ subject to the relations

$$s_i v_l = v_l s_i, \quad l \neq i, i + 1,$$
$$s_i v_i - v_{i+1} s_i = -1, \qquad s_i v_{i+1} - v_i s_i = 1, \tag{20}$$

where the s_i's denote the elementary transpositions in S_n. We will denote this algebra by $\mathcal{H}_n$. We need the following simple lemma.

Lemma 5.3. *Let $A(q)$ be a matrix whose coefficients are differentiate functions in q. We assume $A(q)$ to be diagonalizable in a neighborhood of 1 such that its eigenvalues are of the form q^{n_i}, $i = 1, 2, \ldots,$ with eigenprojections $p_i(q)$, such that $p_i(1)$ is also well-defined. Then $A_i'(1) = \sum_i n_i p_i(1)$.*

The proof of the lemma is straightforward, using that $\sum_i p_i'(q) = \mathrm{d}/\mathrm{d}q(1) = 0$. We now define elements in HB_f^d as follows:

$$x_{i+1} = \lim_{q\to 1}\left(\frac{\mathrm{d}}{\mathrm{d}q}\frac{1}{2}t_{i+1}\right), \quad i = 1, \ldots, f-1, \tag{21}$$

where $t_{i+1} = g_i \cdots g_1 t g_1 \cdots g_i$ and $t_1 = t$. Notice that since $\lim_{q\to 1} t_i = 1$,

$$x_{i+1} = \lim_{q\to 1}\frac{1}{2}\big(g_i' t_i g_i + g_i t_i' g_i + g_i t_i g_i'\big) = \frac{1}{2}\Big(2\tilde{s}_i x_i \tilde{s}_i + \lim_{q\to 1}\big(g_i^2\big)'\Big).$$

It follows from Lemma 5.3 that $\lim_{q\to 1}(g_i^2)' = 2\tilde{s}_i$ (as the eigenvalues of g_i^2 are $q^{\pm 2}$). Hence $x_{i+1} = \tilde{s}_i x_i \tilde{s}_i + \tilde{s}_i$, from which one easily deduces relation (20). These observations imply the following proposition.

Proposition 5.4. *The map $s_i \to \tilde{s}_i$, and $v_i \to x_i$ defines a surjective homomorphism from $\mathcal{H}_f$ onto HB_f^d.*

5.2. *The degenerate Brauer algebra of type B*

Similarly as for the Hecke algebra, we can define a degenerate Brauer algebra of type B as the limit of BB_f as $q \to 1$ when $r' = q^{2n+2m-2}$ and $r = q^{2n-1}$; here e_i will be a multiple of the eigenprojection of g_i for the eigenvalue q^{1-2n} at $q = 1$, and p will be the derivative of t at $q = 1$. The image of the element e_i in $\mathrm{End}(V_{m\epsilon} \otimes V^{\otimes f})$ is equal to E_i (see Section 2.2), which is also well-defined for $q = 1$. It then corresponds to a certain graph in Brauer's centralizer algebra (see [3,29]). It is not hard to check that one obtains the following relations.

Definition 5.5. The degenerate reduced Brauer algebra DB_f^d of type B depending on two parameters m and n is defined via generators $1, p, \tilde{s}_1, \ldots, \tilde{s}_{f-1}, e_1, e_2, \ldots, e_{f-1}$ and relations

(1) e_i's and $\tilde{s}_i$'s satisfy the relations as for the generators of the Brauer algebra (see [3]).

(2) $p\tilde{s}_1 p\tilde{s}_1 + 2p\tilde{s}_1 - 2pe_1 = \tilde{s}_1 p\tilde{s}_1 p + 2\tilde{s}_1 p - 2e_1 p$;

(3) $p^2 = 2(n+m-1)p$;

(4) $e_1 p e_1 = 2n(m+2n-2)e_1$;

(5) $p\tilde{s}_1 p e_1 = (2n-2)pe_1, \qquad e_1 p\tilde{s}_1 p = (2n-2)e_1 p$;

(6) $p\tilde{s}_i = \tilde{s}_i p, \qquad e_i p = pe_i, \quad i > 1$.

Proposition 5.6. *The degenerate Brauer algebra DB_f^d of type B has the same dimension and the same decomposition into simple matrix rings as BB_f.*

Proof. The proof is an adaptation of the analogous proof for BB_f. It is immediate that DB_f^d modulo the ideal I generated by the e_i, $i = 1, 2, \ldots, f-1$, is isomorphic to the degenerate Hecke algebra of type B. One also observes that I itself is isomorphic to Jones' basic construction for $DB_{f-2}^d \subset DB_{f-1}^d$, with respect to the Markov trace (which is also defined in the limit $q \to 1$). One can now proceed as in the proof of Theorem 2.7(a), checking that all the arguments there also work in the limit $q \to 1$. The same also works for the image of DB_{f-1} in $\mathrm{End}_{\mathfrak{so}_{2n}}(V_{m\epsilon} \otimes V^{\otimes f})$ (see the corollary below). We will omit the details. □

Corollary 5.7. *Let ϵ be the highest weight for a spinor representation of $\mathfrak{so}_{2n}$. Then the algebra $\mathrm{End}_{\mathfrak{so}_{2n}}(V_{m\epsilon} \otimes V^{\otimes f})$ is a quotient of the degenerate Brauer algebra of type B. A similar statement holds for $\mathfrak{so}_{2n+1}$ if $m = 1$.*

Nazarov [21] has defined the *degenerate affine Brauer algebra* generated by the Brauer algebra $B(f, N)$ (generated by s_i and $\bar{s}_i$, $1 \leqslant i \leqslant f-1$) along with pairwise commuting elements $y_1, \ldots, y_n$ and central elements $w_1, w_2, \ldots$ subject to the following relations:

$$s_k y_l = y_l s_k, \qquad \bar{s}_k y_l = y_l \bar{s}_k, \qquad l \neq k, k+1, \tag{22}$$

$$s_k y_k - y_{k+1} s_k = \bar{s}_k - 1, \qquad s_k y_{k+1} - y_k s_k = 1 - \bar{s}_k, \tag{23}$$

$$\bar{s}_k (y_k + y_{k+1}) = 0, \qquad (y_k + y_{k+1}) \bar{s}_k = 0, \tag{24}$$

$$\bar{s}_1 y_1^i \bar{s}_1 = w_i \bar{s}_1, \quad i = 1, 2, \ldots. \tag{25}$$

We denote this algebra by $N(f)$. To see how this algebra is connected to DB_f^d, we define elements $x_i \in DB_f^d$ inductively by $x_1 = p$ and $x_{i+1} = \tilde{s}_i x_i \tilde{s}_i + \tilde{s}_i - e_i$.

Proposition 5.8. *The map $s_i \to \tilde{s}_i$, $y_i \to x_i$ and $\bar{s}_i \to e_i$ extends to a homomorphism from Nazarov's degenerate affine Brauer algebra to the degenerate Brauer algebra of type B for the case when $m = 1$ and $w_i = 2n(2n-2)^{i+1}$.*

Proof. The homomorphism property of the map can be checked by direct, but tedious computations. Another way would be to exploit the fact that for $r = q^a$ and $r' = q^b$ for a and b sufficiently large, $BB_f(r, r', q)$ and $DB_f^d(a, b)$ are isomorphic to the centralizers of $U_q\mathfrak{so}_{2n}$ and $\mathfrak{so}_{2n}$, respectively, on $V_{m\epsilon} \otimes V^{\otimes f}$, with m and n depending on a and b. In the following, we will not distinguish in notation between the elements in the abstract algebras and their images in the centralizer algebras. As the R-matrices in these representations can be written down with entries being polynomials in q (e.g. after choosing the special bases of Lusztig or Kashiwara), we have well-defined derivatives for these elements. It is easy to check that $t'(1) = 2p$ and $(g_i^2)'(1) = 2\tilde{s}_i - 2e_i$, using Lemma 5.3. Define inductively $t_1 = t$ and $t_{i+1} = g_i t_i g_i$ for $i > 1$. Then we have $t'(i) = 2x_i$, which follows inductively from

$$t'_{i+1}(1) = 2\tilde{s}_i x_i \tilde{s}_i + \left(g_i^2\right)'(1) = 2x_{i+1}.$$

As the t_i's commute, so do the x_i's (this is easiest checked via representations on the path basis, in which the t_i's act diagonally (see e.g. the discussion of Murphy elements before Proposition 2.6). Now consider the relation in Lemma 4.2(d), i.e. $t_f t_{f+1} e_f = q^{-1} r^{-1} r' e_f$. After substituting $r' = q^{2n+2m-2}$ and $r = q^{2n-1}$, and differentiating with respect to q, we get

$$\begin{aligned}&\left(\frac{\mathrm{d}}{\mathrm{d}q} t_f\right) t_{f+1} e_f + t_f \left(\frac{\mathrm{d}}{\mathrm{d}q} t_{f+1}\right) e_f + t_f t_{f+1} \frac{\mathrm{d}}{\mathrm{d}q}(e_f) \\ &\quad = (2m-2) q^{2m-3} e_f + q^{2m-2} \frac{\mathrm{d}}{\mathrm{d}q} e_f.\end{aligned}$$

Observe that the last summands on each side of the equation cancel each other at $q = 1$. Simplifying the remaining expressions at $q = 1$, we get the equality

$$(x_f + x_{f+1}) e_f = (m-1) e_f,$$

which shows that relations (24) are preserved for $m = 1$. We also observe that by defining relations (3) and (4) of DB_g^d, we have that $e_1 p^i e_1 = 2n(m + 2n - 2) \times (2n + 2m - 2)^i$. This shows that all relations are preserved. $\square$

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